A MULTIVARIATE MAGNITUDE ROBUST
CONTROL CHART
FOR
MEAN SHIFT DETECTION
AND
CHANGE POINT ESTIMATION

THESIS

Ryan M. Harrell, Captain, USAF

AFIT/GOR/ENS/07-09

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

AFIT/GOR/ENS/07-09

A MULTIVARIATE MAGNITUDE ROBUST CONTROL CHART
FOR
MEAN SHIFT DETECTION AND CHANGE POINT ESTIMATION

THESIS

Presented to the Faculty

Department of Operational Sciences

Graduate School of Engineering and Management

Air Force Institute of Technology

Air University

Air Education and Training Command

In Partial Fulfillment of the Requirements for the

Degree of Master of Science in Operations Research

Ryan M. Harrell, BS

Captain, USAF

March 2007

AFIT/GOR/ENS/07-09

A MULTIVARIATE MAGNITUDE ROBUST CONTROL CHART
FOR
MEAN SHIFT DETECTION AND CHANGE POINT ESTIMATION

Ryan M. Harrell, BS
Captain, USAF

Approved:

Marcus B. Perry

Dr. Marcus Perry (Chairman)

14 March 07

date

Sharif Melouk

Dr. Sharif Melouk (Member)

19 Mar 07

date

AFIT/GOR/ENS/07-09

Abstract

Statistical control charts are often used to detect a change in an otherwise stable process. This process may contain several variables affecting process stability. The goal of any control chart is to detect an out-of-control state quickly and provide insight on when the process actually changed. This reduces the off-line time the quality engineer spends assigning causality. In this research, a multivariate magnitude robust chart (MMRC) was developed using a change point model and a likelihood-ratio approach. Here the process is considered in-control until one or more normally distributed process variables permanently and suddenly shifts to out-of-control, stable value. Using average run length (ARL) performance and the relative mean index (RMI), the MMRC is compared to the multivariate cumulative sum (MC1) and the multivariate exponentially weighted moving average (MEWMA). These results show the MMRC performs favorably to the MC1 and MEWMA when the process is initially in-control before shifting out-of-control. Additionally, the MMRC provides an estimate for the change point and out-of-control mean vector. This change point estimator is shown effective for medium to large sudden mean shifts.

AFIT/GOR/ENS/07-09

Dedication

To my wonderfully patient wife

Acknowledgments

Professionally, there are several people I would like to thank for their support and guidance. First off, I'd like to thank my advisor, Dr. Marcus Perry. His candor, patience, guidance and instruction made this research effort possible. Furthermore, I would like to thank Dr. Sharif Melouk for providing insightful feedback. Finally, I would like to thank Drs. James Simpson and Joseph Pignatiello whose research this thesis is based upon.

Personally, I want to thank Brian Crawford, Jon Hudson, Kristen Cavallaro and the rest of the GOR-07M class for the numerous times they distracted me just enough to keep the four foot by six foot window of graduation in front of me. Last but certainly most, I have unending gratitude to my wife, best friend and soulmate. I love you, Chicky.

Ryan M. Harrell

Table of Contents

List of Figures

List of Tables

List of Acronyms and Abbreviations

ARL .... Average Run Length

ARL_0 .... In-control Average Run Length

ASQ .... American Society for Quality

ASQC .... American Society for Quality Control

B .... UCL for the MMRC

CL .... Control Limit

COT .... CUSUM of T

CUSUM .... Cumulative Sum

EMWA .... Exponentially Weighted Moving Average

h .... CL for the MC1

h_4 .... CL for the MEWMA

LCL .... Lower Control Limit

MC1 .... Multivariate CUSUM

MEWMA .... Multivariate EWMA

MLE .... Maximum Likelihood Estimation

MMRC .... Multivariate Magnitude Robust Chart

QE .... Quality Engineer

RMI .... Relative Mean Index

SPC .... Statistical Process Control

UCL .... Upper Control Limit

UMRC .... Univariate Magnitude Robust Chart

A MULTIVARIATE MAGNITUDE ROBUST CONTROL CHART
FOR
MEAN SHIFT DETECTION AND CHANGE POINT ESTIMATION

I. Introduction

1.1 What is Quality?

Everybody has a personal understanding of what quality is, but very few have a concrete definition of it. The cliché, “I'll know quality when I see it,” sums this up nicely. For example, let’s take a look at the automotive industry. I could argue a Mercedes-Benz is a higher quality vehicle because of its bigger engine, leather interior and higher price, and a Toyota is a lower quality vehicle for its smaller engine, cloth interior, and lower price. If this is true, why do both J.D. Power and Associates and Consumer Reports conclude Mercedes-Benz produces poor quality cars? They simply surveyed recent owners about defects and problems and the Toyota owners found fewer problems with their cars. This is the difference between the qualitative and quantitative aspects of quality. While one can like the qualitative feel of leather and surge of power driving a high performance car, one also understand better quality materials do not translate into superior engineering and manufacturing.

In 1925, Walter Shewhart of Bell Laboratories (Bell Labs) was grappling with the same problems. Bell Labs was a joint venture between AT&T and Western Electric to conduct research for both companies. At the time, the phone companies manufactured the phones and leased them to customers. When a phone malfunctioned the phone

company had to fix it at no charge to the consumer. The fewer malfunctioned phones meant the more money the phone company made. Quality control then consisted of testing a phone after manufacture, and fixing the phones with defects. The idea of quality monitoring throughout the manufacturing process did not exist.

Shewhart [16] took his training in statistics and outlined his manifesto on what defines quality in a manufacturing setting. He believed the highest goal in manufacturing was to create variation-free products identical to the engineering specifications. In addition, he broke down the quantitative sources of variability even further into controllable and uncontrollable variables (see Figure 1.1). Most relevant to a quality engineer (QE) are the controllable variables, such as stir rate, pressure and feed rate. By properly setting these variables, a QE can maintain the product or process in statistical control. Unfortunately, known, uncontrollable variables such weather and different batches of raw material can directly affect the variability of a product. Since these variables are known, they are measurable and often a QE can compensate by adjusting the controllable variables. There are factors affecting our process we cannot measure because we are not aware of them, and therefore these factors are unknown. If the correct known variables are incorporated into the process, these unknown variables are assumed to be white noise distributed according to some statistical distribution (*i.e.* Normal). Although these ideas are relatively standard today, they were revolutionary for changing the view of quality in manufacturing.

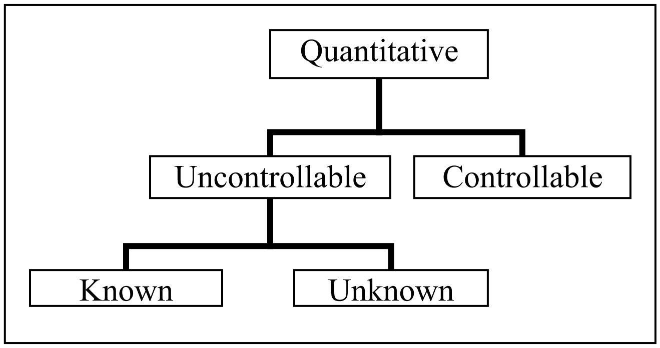

Figure 1.1: Factors Classification

1.2 Phase I

Phase I in statistical process control is an experimental study on the nature of the process you are trying to monitor. A process is defined as something with a desired and measurable target value evolving over time. Process examples range from count data like scratches on a desk to the width of lumber at a mill and intangible data like heart rate and microprocessor switching frequency. These target values are generally either a target mean or target variance, although one can use a target median or a linear/nonlinear model. As a result, this thesis uses the generic term ‘process’ to include all possibilities.

Phase I generally occurs prior to a process is coming online. In this phase the QE’s job is to determine the nature of the process he/she wants to monitor. This is usually accomplished through experimentation and/or the use of historical data.

The preferred method involves a series of designed experiments to understand what factors affect the process, and their statistical distribution. Once the factors are discovered, a response surface study is conducted to optimize the controllable factors to the desired outcome. Since the QE is involved in every step of data collection, he/she receives exactly the data required, and possesses in-depth knowledge of the data

collection techniques and understands the overall data quality. Unfortunately, this process generally involves non-trivial expense of time and money.

If the QE has neither the time or money resources available, he/she can analyze historical data in lieu of experimentation. While this data is usually ‘free’ because it does not cost any additional time or money to collect, the QE often has little to no knowledge about the process used to collect this data. As a result of this limited knowledge, the data quality is automatically in question.

The ideal situation is when the QE has both historical data and the budget to experiment. If the historical data is verified and validated by experimentation, then the QE has a larger data set to work with than just experimentation alone. Clearly, having a large amount of reliable data gives the QE considerable insight into the studied process. Regardless of the data collection method, a particular control chart is selected or developed to best maintain the target value. Then, once Phase I is complete, Phase II monitoring can begin.

1.3 Phase II

In Phase II, the process has completed enough tests and experiments to begin monitoring using a control chart. Here the process is well understood and with each new set of observations, we are trying to answer three questions. The first question is: “Did the process change?” This question then begets two other questions: “If the process did actually change, when did it change and what was the cause?” Basically, just because the chart gives a process out-of-control signal does not mean the QE knows when or why. Unless the chart used has a change point estimator developed for it, he/she is often left

looking to the chart and making an educated guess about the change point. In fact, even with an accurate change point estimator the QE still has to determine causality. This involves investigating the sequence of events, such as a change in raw material supplier or tool wear, causing the chart to signal.

1.3.1 Shewhart $\bar{X}$ Control Chart

The $\bar{X}$ control chart developed by Shewhart [17] monitors whether the process mean is in or out of statistical control. Statistically speaking, this translates into the following hypotheses

$$\begin{aligned} H_0 &: \mu = \mu_0 \\ H_a &: \mu \neq \mu_0 \end{aligned}$$

where μ_0 is the desired in-control mean and μ is the true mean. The null hypothesis, denoted by H_0, states the process is currently in-control and the alternative hypothesis, denoted by H_a, states the process is currently out-of-control. The statistic used to test H_0 versus H_a is the sample subgroup mean

$$\bar{X}_t = \frac{1}{n}\sum_{i=1}^{n} x_{it} \, .$$

Here $\bar{X}_t$ is the average of *n* measurements at time or observation point *t* ranging from one to the most recent observation. Additionally, the $\bar{X}$ control chart assumes all x_{it} values are normally distributed with mean, μ, and variance, σ^2, $\left(x_{it} \sim N\left(\mu, \sigma^2\right)\right)$. Since the observations are normally distributed, the QE can test whether $\bar{X}_t$ at each *t* is within plus or minus *L* standard deviations, σ, of μ $\left(\bar{X}_t \pm L\sigma\right)$. If any $\bar{X}_t > L\sigma$ or

$\bar{X}_t < -L\sigma$, then H_0 is rejected in favor H_a and the process is potentially out-of-control. Essentially, the $\bar{X}$ control chart is a series of sequential hypothesis tests.

How does the QE set the value for *L*? If the data is distributed standard normal, $\left(x_{it} \sim N\left(0,1\right)\right)$, *L* corresponds to the normal distribution inverse of one-half of the probability of declaring a process out-of-control when it is actually in-control. For example, a probability of .0027 corresponds to the standard normal inverse of .00135 resulting in *L* = 3. This equates to a rate of 1/.0027 or an average of one false alarm for every 370 observations. These standard normal inverse values are easily obtainable from any standard statistics text or software package.

For example, Figure 1.2 shows a single run of the $\bar{X}$ control chart. The green dots are from an in-control distribution and the red dots are from an out-of-control process shifted by one positive standard deviation. The brown line is the actual change point, denoted as τ. The dashed line after *t* = 20 indicates the one standard deviation sudden mean shift, or out-of-control value of the mean. The two parallel blue lines are the upper and lower control limits (UCL/LCL) corresponding to $\pm L\sigma$. If the charting statistic exceeds the UCL/LCL, then the chart has signaled indicating a potential out-of-control process. Each dot represents the discrete calculation of the charting statistic.

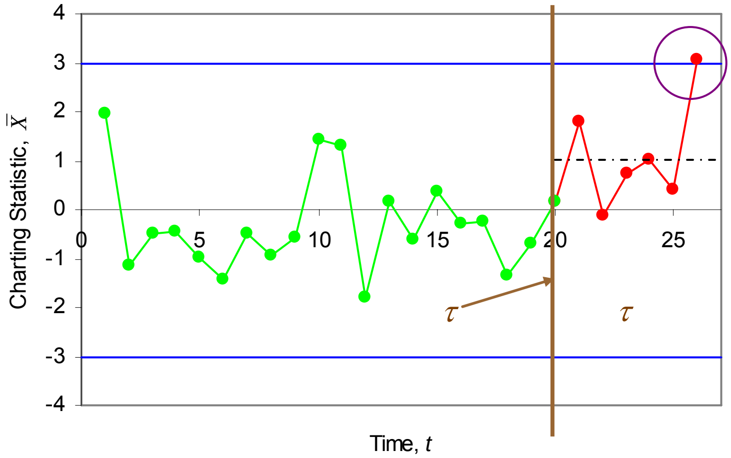

Figure 1.2: $\bar{X}$ Control Chart Example

For this $\bar{X}$ chart, the charting statistic is the standardized sample mean, $\frac{X_t - \hat{\mu}}{\hat{\sigma}}$, where $\hat{\mu}$ and $\hat{\sigma}$ are the estimates obtained from Phase I and the subgroup size $n = 1$. Thus the mean should equal zero and the standard deviation/variance should equal one. As one can see, the green line varies randomly about zero whereas the red line barely goes below zero just once. Finally, the 26th observation exceeds the UCL and the chart signals. The QE then has to answer the questions of when and why. After causality is discovered and the problem is rectified, the chart is restarted and monitoring continues until the next signal.

1.3.2 Average Run Length

The average run length (ARL) is the expected number of observations required until the chart signals. The two types of ARLs are the in-control ARL (ARL_0) and the out-of-control ARL. ARL_0 is the average time to a false alarm when the process remains in-control given a specified UCL/LCL. The out-of-control ARL, on the other hand, is the

time it takes for the chart to signal after a change has occurred. In an ideal situation, ARL_0 is infinity while the out-of-control ARL is the first observation after the change occurred. In reality, false alarms occur in every process, and the in-control and out-of-control ARLs are manipulated on the basis of the UCL/LCL.

In order to compare control charts, ARLs are used. By calibrating all the charts to the same ARL_0, the different charts are compared side by side to determine which signals quicker under different change magnitudes. Thus, competing charts are comparable using an apples to apples approach.

Table 1.1 shows an example ARL comparison between the magnitude robust and the cumulative sum (CUSUM) charts. The quality characteristic in both charts is distributed standard normal and the results are for a sudden mean shift or out-of-control value of the mean. The other details of these charts are unimportant for this example and are discussed in later chapters. The top row of Table 1.1 gives the mean shift on the left and the ARL performance for each chart on the right corresponding to the particular mean shift. Note the mean shift equaling 0.00 is the in-control ARL, ARL_0. The bottom row of the table contains the control limit (CL) for each chart. Unlike the $\overline{X}$ chart, these two charts only have one CL. Notice the CUSUM requires fewer observations to detect for the 1.00 and 1.50 mean shift, but for all other shifts, the magnitude robust takes fewer observations to detect. Unless the ability to detect a mean shift of 1.00 or 1.50 is particularly import to the QE, the magnitude robust chart is the superior chart in terms of ARL performance. This type of evaluation is used to compare many types of charts.

Table 1.1: ARL Comparison Example

mean shift	magnitude robust	CUSUM (k=.5)
0.00	200.00	200.00
0.50	24.73	25.28
1.00	8.28	7.72
1.50	4.42	4.33
2.00	2.87	3.05
2.50	2.09	2.40
3.00	1.63	2.01
3.50	1.36	1.76
4.00	1.18	1.56
	CL = 4.87	CL = 4.00

*from: Pignatiello and Simpson [12]

1.4 Multivariate vs. Multiple Univariate Charts

As systems become more complex, there is a need to monitor more and more variables within a system. The simplest and most straightforward way to accomplish this is to use multiple univariate charts and stop when one of the charts signals. Although this does provide quality control, it does have two major drawbacks. First, the ARL_0 of a group of control charts running in parallel is lower than a single chart because when one chart in the group signals, the whole process signals. This leads to the QE chasing down false alarms more frequently. Consequently, either you have to live with a higher false alarm rate or increase the UCL/LCL width and sacrifice quick detection. Both of these are highly undesirable. Secondly, cross-correlation between variables is not considered when employing several univariate control charts simultaneously. This is a highly dubious omission because variables of a process are often correlated. Both problems are eliminated with multivariate control charts.

1.4.1 Shift Detection Differences

Previous studies show the general superiority of multivariate charts to multiple univariate charts in terms of ARL performance. Take the multivariate CUSUM chart by Pignatiello and Runger [11] versus the multiple univariate chart by Woodall and Ncube [18]. Pignatiello and Runger show their chart is more efficient than the Woodall and Ncube's even though the variables are simulated as independent and the covariance matrix is the identity matrix (see discussion below). As a result, a QE should not equate the performance of a multiple univariate chart to its multivariate extension.

1.4.2 Variable Correlation

To begin, suppose there are two variables to keep in statistical control and the in-control mean is zero, $\boldsymbol{\mu}_0 = [0,0]'$. If we assume no correlation then the distance from the center is:

$$D_e(\boldsymbol{x}) = \sqrt{\sum_i x_i^2} \tag{1.1}$$

where the *e* stands for the Euclidean distance from the center, each x_i is the observation of the i^{th} variable in the process and $\boldsymbol{x}$ is the vector containing the x_i values. Thus, the $\boldsymbol{x}$ vectors $[0,-1]'$, $[-1,0]'$, $\left[\sqrt{2},\sqrt{2}\right]'$ and $\left[\sqrt{2},-\sqrt{2}\right]'$ are all equidistant from $\boldsymbol{\mu}_0$ at a distance equal to one. These distances are depicted as a contour plot in Figure 1.3.

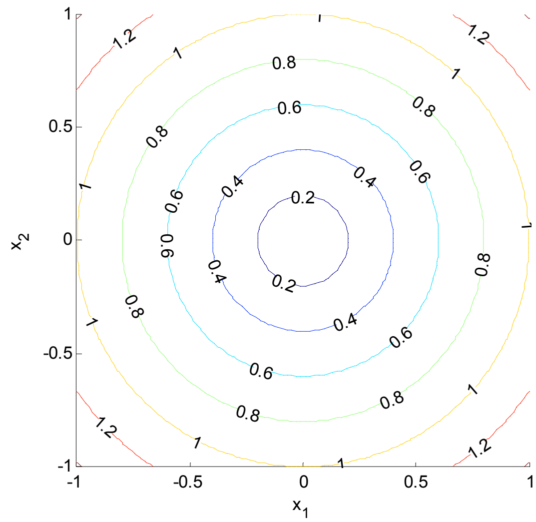

Figure 1.3: Euclidean Distance

If x_1 and x_2 are correlated by the covariance matrix $\Sigma = \begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix}$, then one must use the Mahalanobis distance:

$$D_m = \sqrt{(\boldsymbol{x} - \boldsymbol{\mu}_0)' \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}_0)} \tag{1.2}$$

Note D_e equals D_m when $\boldsymbol{\Sigma}$ is the identity matrix. In this two-variable illustration, D_m simplifies to $\sqrt{x_1^2 + x_2^2 + x_1 x_2}$. When graphed, we obtain the series of ellipsoids in Figure 1.4.

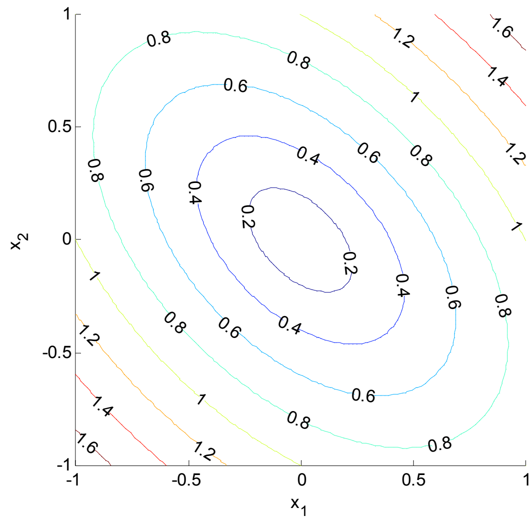

Figure 1.4: Mahalanobis Distance

Since the values are correlated, changes with a like direction have a greater distance, and changes in opposite directions have a shorter distance. Clearly, if you assume Figure 1.3 is true in your charting statistic when Figure 1.4 is reality, then your ARL_0 will vary depending on the mean shift direction. This is known as a directionally variant chart. However if the covariance is included, then the correct distance is calculated regardless of the mean shift direction. This means the chart has the desirable property of directional invariance with a stable, constant ARL_0.

The main disadvantage of using a multivariate chart is the difficulty in finding causality when the chart signals. Namely, the difficulty is in pinpointing the exact out-of-control variable(s) even with an accurate change point estimator. To get around this problem, most QE's run the univariate charts in parallel. Then when the multivariate chart signals, he/she can look at the univariate charts to find the variable(s) causing the signal.

1.5 Problem Definition

Currently, the two multivariate charts with the best ARL performance are the multivariate CUSUM (MC1) developed by Pignatiello and Runger [11] and the multivariate exponentially weighted moving average (MEWMA) chart developed by Lowry *et al.* [8] (described in Sections 3.7.1 and 3.7.2). Both charts are true multivariate charts because they consider cross-correlation by incorporating the covariance matrix. However, the main limitation of these charts is they require tuning to a specific mean shift with the MC1 or narrow range of mean shifts with the MEWMA. Furthermore, neither paper gives an estimate for the change point. The problem becomes to develop a true multivariate chart robust to a wide range of mean shifts, and once the chart signals, provide an estimate for the unknown change point.

1.6 Research Objectives and Assumptions

The objectives of this research are as follows:

1. Derive the multivariate magnitude robust chart (MMRC) by extending the univariate magnitude robust chart (UMRC) developed by Pignatiello and Simpson [12] into the multivariate realm.
2. Using the method of maximum likelihood estimation (MLE), derive change point estimators for the true change point and out-of-control mean vector in the MMRC.
3. Develop a heuristic program and regression equation to provide estimates for MMRC control limits.

4. Use Monte Carlo simulation and the Relative Mean Index (RMI) to compare the MMRC to the MC1 developed by Pignatiello and Runger [11] and MEWMA developed by Lowry *et al.* [8] in terms of ARL performance.
5. Use Monte Carlo simulation to present and evaluate the performance of the derived MMRC change point estimator.

This research effort will investigate the ARL performance of the MMRC, MC1 and MEWMA under the following assumptions:

1. All simulated observations are assumed to be taken from the multivariate normal distribution and have a known or properly estimated in-control mean vector and covariance matrix.
2. The process is assumed to have a sudden shift of the in-control mean vector, also called a step change. In other words, the process has a steady in-control mean vector from zero up to some point in time, τ, where the process mean suddenly shifts to a steady out-of-control mean vector and remains this way until corrected by the QE.

1.7 Thesis Organization

This thesis is divided in to five chapters. Chapter I presented the history and background for SPC with a special emphasis on control charts. Next, the added complexities of multivariate versus multiple univariate charts were discussed and the current problem for this thesis effort was introduced. Chapter II will review relevant literature in univariate control charts, multivariate control charts and comparisons among multiple charts. Chapter III gives the mathematical foundation for the UMRC and

MMRC, and it discusses the MC1 developed by Pignatiello and Runger [11] and the MEWMA developed by Lowry *et al.* [8]. Chapter IV presents the simulation model and the results based on the mathematical foundation derived in Chapter III. Chapter V gives conclusions and recommendations for the QE and presents some areas for future research.

II. Literature Review

2.1 Introduction

Most of the research falls into one of three categories. The first category is univariate SPC where a process is assumed to only have one variable. These are the papers forming the basis of all SPC and as a result, the univariate case is well studied in many top journals (Technometrics, Journal of Quality Technology, etc.). The next category is the extension of univariate SPC into the multivariate realm. Often there are multiple approaches for a single extension. For example, both Crosier [1] and Pignatiello and Runger [11] developed a multivariate extension to the CUSUM control chart. Finally there are published articles summarizing and comparing current multivariate approaches to process monitoring. In these papers, authors pick a particular SPC topic such as ARL performance and compare newer SPC research with older more established SPC research. This chapter reviews all three categories of research in chronological order within each category.

2.2 Univariate Control Charts

Shewhart [16] published a three page paper on the desire of a manufacturer to create uniform items free of variation. Even if the manufacturer can create a uniform product initially, changes in weather, people, raw material, etc. can vary the uniformity of the product. To rectify this problem, he suggested setting tolerance specifications within L standard deviations around a specified target mean. Additionally, he stated without proof, one could calculate the probability of producing a unit within these tolerance specifications about the mean.

After this probability is discussed, he stated, "A constant system of causes always exists after causes which are a function of time have been eliminated". In other words, he suggested once all of the non-random factors are accounted for, the process follows a known probability distribution. Once this distribution is found, a goodness of fit test could be used at each sample observation to test whether this observation follows the known distribution. This statistical test tests two hypotheses. Under the null hypothesis the observations follow the known probability distribution indicating an in-control process, and under the alternative hypothesis the observations do not follow the known probability distribution indicating an out-of-control process. If the test fails the null hypothesis, then the process is considered out-of-control and the QE must find the causality. Ultimately, this paper was the first to suggest applying statistical techniques to quality control.

After six years, Shewhart [17] completed a book on the "why" and "how" of statistical process control. Half of the book is his motivation for quality control. The rest of the book is dedicated to his control chart, now known as the Shewhart $\overline{X}$ control chart. Here Shewhart had the chart signal when the most recent observation or mean of simultaneous observations went beyond a specified number of standard deviation units from a specified sample mean. This simplification allowed a non-statistician to administer and interpret the control chart and report when the chart signals.

In 1954, Page [9] asked and answered the question: Can the previous observations help us create a better chart? In his paper, Page established a new univariate control cart called the cumulative sum (CUSUM), and then later refined it (Page [10]).

Instead of relying solely on the most recent observation, Page's chart used the following two-sided cumulative sum scheme

$$\max\left(S_t^+,0\right) \text{ where } S_t^+ = \sum_{i=1}^{t}\left(x_i - k\right)$$

to detect positive out-of-control shifts of the mean and

$$\min\left(S_t^-,0\right) \text{ where } S_t^- = \sum_{i=1}^{t}\left(x_i + k\right)$$

to detect negative out-of-control shifts of the mean. Here S_t^- and S_t^+ are the CUSUM statistics at time *t*, *k* is a tuning parameter tuned to the mean shift the QE wants to detect, and x_i is the i^{th} observation of the quality characteristic with. If either $S_t^- < h$ or $S_t^+ > h$ occurs, the chart signals indicating a possible out-of-control process. Once the chart signals, the change point estimator is the last time the charting statistic was zero.

Tuned properly, the standard normal CUSUM (*i.e.* $k = 0.50$, $h = 4$) chart detects smaller mean shifts more quickly than the Shewhart $\overline{\mathrm{X}}$ control chart, but is slower to detect large mean shifts of two standard deviations or greater. Unfortunately, most of the time the QE will have no knowledge about this mean shift. As a result, if one incorrectly calibrates the CUSUM, then the small shift detection capabilities can become degraded. Even with these shortcomings, the CUSUM is widely regarded as a significant advancement in the field of control charts.

Roberts [13] later came up with a different approach to using previous data, the geometric moving average chart. This approach is so called because he used a geometric moving average to develop the charting statistic:

$$Z_i\left(r\right) = r\overline{X}_i + (1-r)Z_{i-1}\left(r\right) \tag{2.1}$$

where r is the amount between zero and one and the i^{th} ($i = \{0,1,\ldots n\}$) subgroup mean, $\overline{X}_i$, is weighted against the previous Z_i calculation, Z_{i-1}. Furthermore, Z_0 is the in-control mean, n is the most recent observation and the chart signals when the first Z_n exceeds the CL. Note the Shewhart $\overline{X}$ chart is a special case of the EWMA chart when $r = 1$. To illustrate the chart's effectiveness with regards to change detection, Roberts studied ARL performance for different values of r. While a completely different approach than the CUSUM, this chart is well known to provide strikingly similar performance. Over time, the chart's name changed to the exponentially weighted moving average (EWMA) chart, and the chart is commonly used today.

The fourth major chart is the UMRC by Pignatiello and Simpson [12]. Unlike the CUSUM and EWMA, the UMRC does not use a weighted or cumulative sum in its calculation. Instead, it is based on the family of generalized likelihood-ratio (GLR) tests. The GLR uses the ratio of the likelihood under H_a over the likelihood under H_0 where H_a states the process is in-control and H_0 states the process is out of control. If this GLR test exceeds some defined threshold, H_0 is rejected. The term generalized in the GLR means some of the input variables are unknown and estimated using MLE. For SPC control charts, these unknown input variables are usually the in- and out-of-control variance and mean and the change point. The paper by Hawkins, Qiu and Kang [2] (reviewed later in this chapter) assumes the in- and out-of-control variance and mean and the change point are unknown. However, the UMRC is less general because Pignatiello and Simpson assumed only the out-of-control mean and the change are unknown. Additionally, they included a change point model into their chart. The change point model assumes the

process mean is in-control up to a point, denoted by τ, when the process mean shifts some time between τ and $\tau+1$ with $\tau+1$ as the first observation of the out-of-control process.

In their paper, Pignatiello and Simpson use Monte Carlo simulation to compare the ARL performance of the UMRC to the CUSUM. They show the ARL performance of the UMRC is superior any one CUSUM chart over all tested mean shift magnitudes even though an accurately tuned CUSUM chart is superior to the UMRC. Additionally, Pignatiello and Simpson's chart also provides the MLE for the time of the step change. This is in contrast to the $\overline{\mathrm{X}}$ where the quality control engineer has to look at the chart and take an educated guess on when the change occurred. Having an estimate for the process change point saves expensive down time as a result of searching for and correcting the source of the process change. The main disadvantage to the UMRC is the increased level of computation because the log-likelihood-ratio statistic is evaluated for all possible change points. However, a simple computer program on a modest computer can easily perform the required calculations. The UMRC is discussed in Section 3.2.

Using generalized likelihood-ratios, Hawkins, Qiu and Kang [2] proposed a univariate unknown parameter change point model. Like Pignatiello and Simpson [12], they assumed a change point model. However, the difference is the process in-control and out-of-control means and the process variance are assumed unknown by Hawkins, Qiu and Kang. As a result, the traditional sequence of plugging Phase I values into a Phase II process monitoring control chart is not needed. In fact, they stated monitoring can begin on the third observation. However, they pointed out three observations will contain a lot of variance and is not enough data to validate the normality assumptions the

chart is based on. They recommended running the process until the variance stabilizes and the normality assumption is verifiable. Thus, they presented their control chart as a seamless transition from Phase I to Phase II.

To test the null hypothesis the process is in-control versus the alternative hypothesis the process is out-of-control, Hawkins, Qiu, and Kang used a two-sample *t*-test statistic, denoted as $T_{j,n}$, where *j* is the candidate change point and *n* is the most recent observation. Then $T_{j,n}$ is maximized over all possible *j* from one to *n* - 1 to obtain the maximum separation between the in-control and out-of-control means, and this maximum is denoted as $T_{\max,n}$. The chart signals when $T_{\max,n}$ exceeds the appropriate CL, denoted by h_n, where the value of h_n is dependent on the value of *n*. Since there was no closed-form solution for h_n, the paper provided both a closed-form approximation and tables obtained through simulation for h_n. Once the chart signals, the *j* maximizing $T_{j,n}$ is the MLE for the true change point.

To counter the need for tuning in the EWMA chart developed by Roberts [13], Han and Tsung [3] proposed the generalized EWMA control chart (GEWMA). This was accomplished by modifying the EWMA to:

$$T_{GE}\left(c\right) = \inf\left\{ n \geq 1 : \max_{1 \leq k \leq n} \left| \overline{W}_n\left(\frac{1}{k}\right) \right| \geq c \right\}$$

Here $T_{GE}\left(c\right)$ represents the time the chart signals for the infinium of the n^{th} observation where the absolute value of charting statistic, $\overline{W}_n\left(\frac{1}{k}\right)$, maximized over all values of *k* (*k* = {1,2,…,n}) exceeds the CL *c*. $\overline{W}_n\left(\frac{1}{k}\right)$ is defined as

$$\overline{W}_n\left(\frac{1}{k}\right)=\frac{\sqrt{(2-1/k)}}{\sqrt{1/k\left[1-(1-1/k)^{2n}\right]}}W_n\left(\frac{1}{k}\right),$$

$$W_n\left(\frac{1}{k}\right)=\frac{1}{k}X_n+\left[1-\frac{1}{k}\right]W_{n-1}\left(\frac{1}{k}\right)$$

where $\frac{\sqrt{(2-1/k)}}{\sqrt{1/k\left[1-(1-1/k)^{2n}\right]}}$ is the standard deviation of $W_n\left(\frac{1}{k}\right)$. This $W_n\left(\frac{1}{k}\right)$ is the same equation as Robert's EMWA in **(2.1)** except Han and Tsung only considered individual observations, not subgroup means. In their paper, Han and Tsung used Monte Carlo simulation to show the ARL performance of the GEWMA is superior to the optimal EWMA developed by Wu [19] for all mean shifts and the CUSUM developed by Page [9] for any mean shift not between the interval (0.7842δ, 1.3798δ) where δ is the predicted mean shift. Essentially, the authors took the EWMA and made it robust to a wider array of potential mean shifts.

2.3 Multivariate Control Charts

Hotelling [5] developed the first multivariate chart in 1947 to improve the testing of bombsights. Using Shewhart's $\overline{\mathrm{X}}$ control chart as a basis, Hotelling modified the chart to allow for a vector of observations. Additionally, he recognized the possibility of correlated quality measures, and therefore included a covariance matrix into his T^2 statistic. The use of T in the T^2 statistic is significant because it the square of the student-t statistic. Like the $\overline{\mathrm{X}}$ control chart, the T^2 chart signaled when the most recent T^2 calculation went beyond some user specified upper or lower control limit. The drawbacks to the T^2 control chart were twofold. First off, it suffered the same poor ARL

performance for detecting small shifts as the univariate $\overline{X}$ control chart. Secondly, the T^2 statistic was more computationally intensive with matrix multiplication, matrix inversion, and the required development of the covariance matrix. Although now a trivial issue with computers, these computational issues were a big hindrance to adaptation in 1947.

In 1985, Woodall and Ncube [18] created the first multivariate CUSUM control chart entitled the MCUSUM chart. Although called MCUSUM, the chart actually consists of multiple univariate CUSUM charts run simultaneously. Each of these CUSUM charts independently monitors each variable within a process. As the process runs, the first of these independent CUSUM charts to signal causes the MCUSUM to signal. Using a Markov chain approach, they showed the MCUSUM is often superior to Hotelling's T^2 chart when the monitored quality characteristic is a bivariate normal random variable. The obvious disadvantage of Woodall and Ncube's chart is the more univariate CUSUMs implemented, the worse the ARL performance becomes. Additionally, the MCUSUM chart suffers the same tuning issues as the univariate CUSUM, and cannot incorporate correlation between process variables. Most importantly, Woodall and Ncube's article started a renewed interest in multivariate charts.

Inspired by Woodall and Ncube, Crosier [1] created two true multivariate CUSUM charts. The first chart takes the square root of Hotelling's T^2 statistic and used this to generate a univariate CUSUM chart. He entitles this the COT or CUSUM of T. The second statistic calculates directly from the vector of observations. Unfortunately, Crosier uses a limited number of runs (≤ 400) in his Monte Carlo studies to come up with

his results. Despite this, Crosier showed the direct multivariate CUSUM chart yields superior ARL performance relative to both the COT and MCUSUM.

Around the same time, Pignatiello and Runger [11] created their own pair of multivariate CUSUM charts, entitled the MC1 and MC2. Although they took a different approach than Crosier [1], Pignatiello and Runger discovered the MC1 produces similar results to Crosier's direct multivariate CUSUM. Unlike Crosier, Pignatiello and Runger decided to employ a run size of 6,000 in their Monte Carlo experiments to generate more accurate results. The MC2 chart, while better than Hotelling's T^2 chart, produces inferior ARL performance to Woodall and Ncube [18]. Additionally, the paper contains a lengthy discussion on the subject of directional invariance, detailing how directional invariance maintains consistent ARLs, but hinders diagnosis efforts when the chart signals. The MC1 charting statistic is discussed later in Section 3.7.1.

After the multivariate CUSUMs were developed by Woodall and Ncube [18], Crosier [1] and Pignatiello and Runger [11], Lowry *et al.* [8] extended the EWMA into the multivariate EWMA, denoted by them as the MEWMA. Unlike the multivariate CUSUM schemes, the MEWMA is a direct extension from the univariate case. By incorporating the covariance matrix into its charting statistic calculation, the MEWMA allowed for correlation among variables in a process. Lowry *et al*. actually created two charts based on the derivation of the covariance matrix. One approach calculated the exact covariance, while the other approach used the steady-state covariance. Of the two variance calculations, Lowry *et al*. showed in their results the superiority of using the exact covariance matrix with regards to ARL performance. Like Pignatiello and Runger, Lowry *et al*. used a Monte Carlo simulation with the same run size of 6,000 to evaluate

the ARL performance. They concluded their chart is superior, in terms of ARL performance, to Hotelling's T^2 and the MCUSUM and comparable to Crosier [1] and the MC1. The MEWMA is discussed later in Section 3.7.2.

An untested parameter in an otherwise comprehensive paper was their choice of the simulated change point. They stated the process is likely to be in-control for a while and then go out-of-control, yet they started all of their simulations as out-of-control from time zero. As this thesis shows in Section 4.7, the choice of control limit and the resulting in-control ARL is dependent on when the change point actually occurs.

Runger and Prabhu [14] (1992) used a Markov chain approach to evaluate the ARL of the MEWMA control chart. Using symmetry and orthogonal invariance, they generate results within 4% of Lowry *et al*. Since their approach was analytical, none of the error associated with the Monte Carlo simulations is present.

Within the last year, Zamba and Hawkins [20] developed a multivariate extension to Hawkins, Qiu and Kang [2]. Like Hawkins, Qiu and Kang, they made no assumptions about in- and out-of-control mean values and variance. The method proposed by these authors essentially eliminated the Phase I study required by traditional approaches. In order to accomplish this, Zamba and Hawkins split the n observations into two subgroups. Then using maximum likelihood estimation to find the maximum mean distance between all possible paired subgroups, they calculated a T^2 statistic. Let n equal the number of observations, p equal the number of variables and α equal the specified false alarm rate. If the T^2 statistic is beyond a specified $h_{n,p,\alpha}$, then the chart signals. Differing from other charts, $h_{n,p,\alpha}$ changes over time as a function of n. Since there is no

closed-form solution for the control limits $h_{n,p,\alpha}$, for n = 1,2,…, Zamba and Hawkins obtained these values using simulation.

While this chart could readily replace other charts, the authors pointed out the existence of situations where a complete Phase I study is necessary and the use of their chart as a supplement. In addition to mean shift detection, their model also gave an estimate of the change point τ. Overall, this paper was a welcome addition to multivariate statistical process control, which this thesis is a special case of Zamba and Hawkins' paper.

2.4 Overview Evaluations

For univariate charts, Lai [6] provided a theoretical overview of change point estimation in-control charts developed up to 1995. The lack of a built-in change point estimator in the $\overline{X}$ and other charts presented a major challenge. Lai surveyed all attempts up to publication and combined them into a unified theory. While Lai's purpose was to present and prove his theory, he presented an extensive bibliography and discussion of existing charts.

In 1995, Lowry and Montgomery [7] presented a review of the state of the art in multivariate control charting. Their paper includes a thorough overview of the aforementioned multivariate papers with the exception of Zamba and Hawkins [20], since their paper was not yet published. Some discussion is provided on each control chart from an implementation point of view. In conducting their review, Lowry and Montgomery pointed out a couple of areas for future research. The first is the difficulty in interpreting out-of-control signals, and the second is the lack of multivariate charts for

monitoring auto correlated data. Like Lowry and Montgomery, this thesis will compare the ARL performance of the multivariate magnitude robust chart to the MC1 and MEWMA.

2.5 Conclusion

This chapter presented an overview of some relevant past and present papers in SPC control charts. For univariate charts, both the canonical charts like Page's CUSUM and Robert's EWMA and more recent advances like the GEWMA developed by Han and Tsung were reviewed. Moreover, the review showed the research evolution from considering only the most recent observation in Shewhart's $\overline{X}$ control chart to using a likelihood-ratio test in Pignatiello and Simpson's magnitude robust control chart. Then these univariate charts were extended into multivariate space with the exception of Pignatiello and Simpson's control chart. Zamba and Hawkins proposed a generalized approach to the magnitude robust in the multivariate realm, but they do not consider the case where the in-control mean and covariance matrix is known. As a result, this thesis extends the univariate magnitude robust control chart of Pignatiello and Simpson, which is a special case of Zamba and Hawkin's method.

Several of these control charts are interrelated. First off, Zamba and Hawkin's paper is a generalization of the paper by Hawkins, Qiu, and Kang. Pignatiello and Simpson's paper is a special case of the method described in Hawkins, Qiu, and Kang's paper. Again, this thesis is a special case of the method developed by Zamba and Hawkins. Additionally, Pignatiello and Simpson's paper is a special case of this thesis.

III. Methodology

3.1 Introduction

This thesis will fill the gap between the Pignatiello and Simpson [12] UMRC and the Zamba and Hawkins [20] unknown parameter change point chart. In this chapter, the UMRC is directly extended into the multivariate realm. Unlike Zamba and Hawkins, this chart will assume a known in-control mean and covariance matrix, which requires a Phase I study. In other words, the MMRC is a special case of the Zamba and Hawkins method.

To develop the MMRC, the UMRC is derived first and then followed by the derivation of the MMRC. In the interest of brevity, these derivations will leave out some intermediate steps. For those interested in the algebraic details of the derivation, refer to Appendix A. These derivations are highly similar because they both involve a likelihood-ratio test and MLE. After deriving both charts, a single run of the MMRC is presented and explained. To wrap up the MMRC, sections 3.5 and 3.6 will deal with the issue of specifying the value for the control limit (CL) and the handling of false alarms.

Since the ARL performance of the MMRC is compared to the ARL performance of the MC1 and MEWMA in Chapter IV, both the MC1 and MEWMA are discussed in this chapter. Finally, the RMI by Han and Tsung [4] is introduced. RMI aids the QE in determining the chart possessing superior ARL performance over a range of mean shift magnitudes.

3.2 The Univariate Magnitude Robust Chart

The UMRC is based on a change point model and the likelihood-ratio test and was developed by Pignatiello and Simpson [12]. This means the process is assumed in-control up until the change point τ where the process has a sudden shift in the mean between τ and $\tau+1$. As a result, the first observation sampled from the changed process is obtained at $\tau+1$. Throughout this derivation and the derivation in Section 3.3, the index variable denoted by *t* always refers to a discrete point in time ranging from one to the most recent observation *T*. Moreover, the index variable *c* is a candidate change point ranging from zero to the observation *T* - 1. The likelihood-ratio is used to test the hypotheses

$$\begin{aligned} &H_0 : \mu_t = \mu_0 \text{ for } 1 \le t \le T \\ &H_a : \mu_t = \mu_0 \text{ for } 1 \le t \le T \text{ and } \mu_t = \mu_a \text{ for } \tau+1 \le t \le T \end{aligned} \tag{3.1}$$

where μ_0 is the in-control mean and μ_a is the out-of-control mean. Both μ_a and τ are assumed unknown. Additionally, the UMRC assumes the observations are independent and normally distributed.

To begin, assume the random variable x represents observations from the process. These observations are normally distributed with the density given by

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \tag{3.2}$$

where μ and σ are the parameter mean and standard deviation respectively. Next, the likelihood function is formed under the distribution specified by H_0:

$$L_0(\bar{\boldsymbol{x}}) = \prod_{t=1}^{T}\frac{1}{\sqrt{2\pi}\sigma_{\bar{x}}}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}(\bar{x}_t-\mu_0)^2\right)$$

As you can see x and σ from the Equation **(3.2)** were replaced with parameter estimates $\bar{x}_t$ and $\sigma_{\bar{x}}$. The bar above the x represents the possibility of using subgroup averages at each *t* point in time. In other words, over time you should have a vector of means $\bar{\boldsymbol{x}} = [\bar{x}_1, \bar{x}_2, ..., \bar{x}_T]$ up to the most recent observation *T*. The likelihood function under the distribution specified by H_a is given by

$$L_a(\tau \mid \bar{\boldsymbol{x}}) = \prod_{t=1}^{\tau} \frac{1}{\sqrt{2\pi}\sigma_{\bar{x}}} \exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}(\bar{x}_t - \mu_0)^2\right) \prod_{t=\tau+1}^{T} \frac{1}{\sqrt{2\pi}\sigma_{\bar{x}}} \exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}(\bar{x}_t - \mu_a)^2\right)$$

Taking the ratio of L_a to L_0:

$$\frac{L_a(\tau \mid \bar{\boldsymbol{x}})}{L_0(\bar{\boldsymbol{x}})} = \frac{\prod_{t=\tau+1}^{T} \exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}(\bar{x}_t - \mu_a)^2\right)}{\prod_{t=\tau+1}^{T} \exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}(\bar{x}_t - \mu_0)^2\right)} \tag{3.3}$$

From this point one could theoretically use the ratio L_a to L_0 as defined. However, a natural log transformation simplifies the math:

$$R(\tau \mid \bar{\boldsymbol{x}}) = \log_e \frac{L_a(\tau \mid \bar{\boldsymbol{x}})}{L_0(\bar{\boldsymbol{x}})} = \frac{1}{2\sigma_{\bar{x}}^2}\left(\sum_{t=\tau+1}^{T}(\bar{x}_t - \mu_0)^2 - \sum_{t=\tau+1}^{T}(\bar{x}_t - \mu_a)^2\right) \tag{3.4}$$

Clearly, *R* is greater when the subgroup $\bar{x}_t$ is closer to the alternative mean μ_a due to an increase in $\sum_{t=\tau+1}^{T}(\bar{x}_t - \mu_0)^2$. Thus, a greater *R* indicates we are more likely to reject H_0.

Even with a likelihood-ratio test in place, the problem of finding estimates for μ_a and τ still exist. Although the true values for μ_a and τ are assumed unknown, they are estimable. The MLE for μ_a given τ is

$$\hat{\mu}_a(\tau) = \bar{\bar{x}}_{T,\tau} = \frac{1}{T-\tau}\sum_{t=\tau+1}^{T}\bar{x}_t \,. \tag{3.5}$$

Essentially $\hat{\mu}_a(\tau)$ is the overall mean for the $T-\tau$ most recent subgroups. Substituting $\bar{\bar{x}}_{T,\tau}$ into **(3.4)** and simplifying:

$$\begin{aligned} R(\tau \mid \bar{\boldsymbol{x}}) &= \frac{1}{2\sigma_{\bar{x}}^2}\left(\sum_{t=\tau+1}^{T}(\bar{x}_t-\mu_0)^2 - \sum_{t=\tau+1}^{T}(\bar{x}_t-\bar{\bar{x}}_{T,\tau})^2\right) \\ &= \frac{T-\tau}{2\sigma_{\bar{x}}^2}(\bar{\bar{x}}_{T,\tau}-\mu_0)^2 \end{aligned} \tag{3.6}$$

This derivation is the MLE for $\hat{\mu}_a(\tau)$ and its use further simplifies *R*. The task now is to maximize Equation **(3.6)** by evaluating it over all possible values of *c*:

$$R_T = R(\tau \mid \bar{\boldsymbol{x}}) = \max_{0\le c<T}\frac{T-c}{2\sigma_{\bar{x}}^2}(\bar{\bar{x}}_{T,c}-\mu_0)^2 \,. \tag{3.7}$$

Here $\hat{\tau}$ is the value of τ maximizing $R(\tau \mid \bar{\boldsymbol{x}})$ in Equation **(3.6)**.

Equation **(3.7)** is equivalent to the original function in **(3.3)** with a log transformation and an estimate for μ_a maximized over the entire range of τ. When you implement R_T as a control chart, the process is assumed in-control until R_T exceeds some CL *B* ($R_T > B$). At this point, H_0 is rejected in favor of H_a from the hypotheses in **(3.1)**.

Assuming the chart has signaled, an estimate for τ is needed. This estimate is the same $\hat{\tau}$ from **(3.7)**. When computed separately,

$$\hat{\tau} = \arg\max_{0\le c<T}\frac{T-c}{2\sigma_{\bar{x}}^2}(\bar{\bar{x}}_{T,c}-\mu_0)^2 \,. \tag{3.8}$$

Next we substitute $\hat{\tau}$ for τ into **(3.5)**:

$$\hat{\mu}_a\left(\hat{\tau}\right) = \bar{\bar{x}}_{T,\hat{\tau}} = \frac{T-\hat{\tau}}{2\sigma_{\bar{x}}^2}\sum_{t=\hat{\tau}+1}^{T}\bar{x}_t \, . \qquad \textbf{(3.9)}$$

Both point estimates, $\hat{\tau}$ and $\hat{\mu}_a$, are maximum likelihood estimators for τ and μ_a.

Currently, no closed-form solution for finding *B* exists. Pignatiello and Simpson use Monte Carlo simulation to obtain in-control ARL values for specific levels of *B* in Table 3.1.

B	ARL
4.00	78.626
4.25	97.282
4.50	123.666
4.75	152.392
4.87	167.626
5.00	187.604
5.25	232.366
5.50	292.361
5.75	357.632
6.00	457.914

Table 3.1: ARL values from Pignatiello and Simpson [12], Table 4

Applying a log transformation and using ordinary least squares on the data shown in Table 3.1 generates a regression estimate for *B* based on a desired ARL_0:

$$\hat{B} = \frac{\log_e \mathrm{ARL}_0 - 0.8728}{.8732} \qquad \textbf{(3.10)}$$

By inputting an ARL_0 ranging from 78.626 to 457.914, a UMRC implementer can obtain an appropriate CL estimate, $\hat{B}$.

Implementing the UMRC is fairly straightforward. First, the QE needs to select an appropriate *B* value from Equation **(3.10)**. Then as observations come in, compute R_T using Equation **(3.7)**. Once the chart signals the value of τ maximizing R_T becomes $\hat{\tau}$ and is equivalent to Equation **(3.8)**. As a result, there is no need to explicitly calculate

(3.8). Now input $\hat{\tau}$ into Equation **(3.9)** to obtain an MLE for $\hat{\mu}_a$ given $\hat{\tau}$. Repeat this process at every chart signal and the chart is fully implemented.

3.3 The Multivariate Magnitude Robust Chart

Having derived the UMRC, this chart is now extended to the multivariate case using a similar process with *p* variables where τ, *c*, *t* and *T* are as previously defined. Again, a multivariate normal distribution is assumed for random variable vector ***x*** with the following probability density function

$$f\left(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{S}\right)=\frac{1}{\left(2\pi\right)^{p/2}\left|\boldsymbol{S}\right|^{1/2}}\exp\left\{-\frac{1}{2}\left(\boldsymbol{x}-\boldsymbol{\mu}\right)'\boldsymbol{S}^{-1}\left(\boldsymbol{x}-\boldsymbol{\mu}\right)\right\} \tag{3.11}$$

where $\boldsymbol{x}'=\left[x_1,x_2,...,x_p\right]$ is a 1 by *p* vector of observations, $\boldsymbol{\mu}'=\left[\mu_1,\mu_2,...,\mu_p\right]$ is a 1 by *p* vector of means and ***S*** is a *p* by *p* covariance matrix. In the literature, ***S*** is generally symbolized by $\boldsymbol{\Sigma}$, but to reduce confusion with the summation symbol, ***S*** is used instead.

The MMRC has the same hypothesis test as **(3.1)**; however, parameter vectors are considered instead of scalars:

$$\begin{aligned}&H_0:\boldsymbol{\mu}_t=\boldsymbol{\mu_0}\ \text{ for } 1\le t\le T\\&H_a:\boldsymbol{\mu}_t=\boldsymbol{\mu_0}\ \text{ for } 1\le t\le \tau \ \text{ and } \boldsymbol{\mu_t}=\boldsymbol{\mu_a}\ \text{ for } \tau+1\le t\le T\end{aligned} \tag{3.12}$$

Like the UMRC, the MMRC employs a change point model and likelihood-ratio test to derive the test statistic. Using **(3.11)**, the likelihood function under H_0 is

$$L_0\left(\boldsymbol{X}\right)=\prod_{t=1}^{T}\frac{1}{\left(2\pi\right)^{p/2}\left|\boldsymbol{S}\right|^{1/2}}\exp\left\{-\frac{1}{2}\left(\boldsymbol{x}_t-\boldsymbol{\mu_0}\right)'\boldsymbol{S}^{-1}\left(\boldsymbol{x}_t-\boldsymbol{\mu_0}\right)\right\}$$

where ***X*** is the matrix containing the $\boldsymbol{x}_t$ vectors. Under H_a, the likelihood function is

$$L_a\left(\tau,\boldsymbol{\mu_a} \mid \boldsymbol{X}\right)=\prod_{t=1}^{\tau}\frac{1}{\left(2\pi\right)^{p/2}\left|\boldsymbol{S}\right|^{1/2}}\exp\left\{-\frac{1}{2}\left(\boldsymbol{x}_t-\boldsymbol{\mu_0}\right)'\boldsymbol{S}^{-1}\left(\boldsymbol{x}_t-\boldsymbol{\mu_0}\right)\right\}$$
$$\prod_{t=\tau+1}^{T}\frac{1}{\left(2\pi\right)^{p/2}\left|\boldsymbol{S}\right|^{1/2}}\exp\left\{-\frac{1}{2}\left(\boldsymbol{x}_t-\boldsymbol{\mu_a}\right)'\boldsymbol{S}^{-1}\left(\boldsymbol{x}_t-\boldsymbol{\mu_a}\right)\right\}$$

Now take the ratio of L_a to L_0:

$$\frac{L_a\left(\tau,\boldsymbol{\mu_a} \mid \boldsymbol{X}\right)}{L_0\left(\boldsymbol{X}\right)}=\frac{\prod_{t=\tau+1}^{T}\frac{1}{\left(2\pi\right)^{p/2}\left|\boldsymbol{S}\right|^{1/2}}\exp\left\{-\frac{1}{2}\left(\boldsymbol{x}_t-\boldsymbol{\mu_a}\right)'\boldsymbol{S}^{-1}\left(\boldsymbol{x}_t-\boldsymbol{\mu_a}\right)\right\}}{\prod_{t=\tau+1}^{T}\frac{1}{\left(2\pi\right)^{p/2}\left|\boldsymbol{S}\right|^{1/2}}\exp\left\{-\frac{1}{2}\left(\boldsymbol{x}_t-\boldsymbol{\mu_0}\right)'\boldsymbol{S}^{-1}\left(\boldsymbol{x}_t-\boldsymbol{\mu_0}\right)\right\}} \tag{3.13}$$

Equation **(3.13)** is the multivariate equivalent of **(3.3)**. Likewise, this function is simplified using a log transformation:

$$R\left(\tau \mid \boldsymbol{X}\right)=\log_e\frac{L_a}{L_0}=\frac{1}{2}\sum_{t=\tau+1}^{T}\left[\left(\boldsymbol{x}_t-\boldsymbol{\mu_0}\right)'\boldsymbol{S}^{-1}\left(\boldsymbol{x}_t-\boldsymbol{\mu_0}\right)-\left(\boldsymbol{x}_t-\boldsymbol{\mu_0}\right)'\boldsymbol{S}^{-1}\left(\boldsymbol{x}_t-\boldsymbol{\mu_0}\right)\right]$$
$$=\frac{1}{2}\left[\left(T-\tau\right)\boldsymbol{\mu_0}'\boldsymbol{S}^{-1}\boldsymbol{\mu_0}-2\boldsymbol{\mu_0}'\boldsymbol{S}^{-1}\sum_{t=\tau+1}^{T}\boldsymbol{x}_t+2\boldsymbol{\mu_a}'\boldsymbol{S}^{-1}\sum_{t=\tau+1}^{T}\boldsymbol{x}_t-\left(T-\tau\right)\boldsymbol{\mu_a}'\boldsymbol{S}^{-1}\boldsymbol{\mu_a}\right]$$

Again, an estimate for $\boldsymbol{\mu_a}$ is needed. In order to obtain a maximum likelihood estimator for $\boldsymbol{\mu_a}$, given any τ, we take the partial derivative of *R* with respect to $\boldsymbol{\mu_a}$:

$$\frac{\partial R}{\partial\,\boldsymbol{\mu_a}}=\boldsymbol{S}^{-1}\sum_{t=\tau+1}^{T}\boldsymbol{x}_t-\left(T-\tau\right)\boldsymbol{S}^{-1}\boldsymbol{\mu_a}\,. \tag{3.14}$$

Setting this equal to zero yields:

$$\boldsymbol{S}^{-1}\sum_{t=\tau+1}^{T}\boldsymbol{x}_t-\left(T-\tau\right)\boldsymbol{S}^{-1}\boldsymbol{\mu_a}=0$$
$$\hat{\boldsymbol{\mu}}_{\boldsymbol{a}}\left(\tau\right)=\frac{1}{T-\tau}\sum_{t=\tau+1}^{T}\boldsymbol{x}_t=\overline{\boldsymbol{x}_{T,\tau}}\,. \tag{3.15}$$

Thus $\overline{\boldsymbol{x}_{T,\tau}}$ is the MLE for $\boldsymbol{\mu_a}$ given τ. Substituting $\overline{\boldsymbol{x}_{T,\tau}}$ in for $\boldsymbol{\mu_a}$ in *R* and simplifying:

$$R(\tau \mid \boldsymbol{X}) = \frac{T-\tau}{2}\left[\left(\boldsymbol{\mu_0} - \overline{\boldsymbol{x}_{T,\tau}}\right)' \boldsymbol{S}^{-1}\left(\boldsymbol{\mu_0} - \overline{\boldsymbol{x}_{T,\tau}}\right)\right]. \quad \textbf{(3.16)}$$

As with Equation **(3.7)**, the MMRC charting statistic is found by maximizing *R* over all possible values of *c*:

$$R_{\max} = R(\tau \mid \boldsymbol{X}) = \max_{0 \le c < T} \frac{T-c}{2}\left[\left(\boldsymbol{\mu_0} - \overline{\boldsymbol{x}_{T,c}}\right)' \boldsymbol{S}^{-1}\left(\boldsymbol{\mu_0} - \overline{\boldsymbol{x}_{T,c}}\right)\right] \quad \textbf{(3.17)}$$

where $\hat{\tau}$ is the value of τ maximizing $R(\tau \mid \boldsymbol{X})$ in Equation **(3.16)**. When implemented, the process is assumed in-control until $R_{\max}$ exceeds a specified CL *B* ($R_{\max} > B$). Section 4.6 gives a regression based approach for estimating *B* for specific ARL_0 values. Once the chart signals, H_0 is rejected in favor of H_a from the hypotheses in **(3.12)**.

After the chart signals, an estimate for τ is computed using:

$$\hat{\tau} = \arg\max_{0 \le c < T} \frac{T-c}{2}\left[\left(\boldsymbol{\mu_0} - \overline{\boldsymbol{x}_{T,c}}\right)' \boldsymbol{S}^{-1}\left(\boldsymbol{\mu_0} - \overline{\boldsymbol{x}_{T,c}}\right)\right] \quad \textbf{(3.18)}$$

Moreover, this $\hat{\tau}$ is the same $\hat{\tau}$ from Equation **(3.17)**. Substituting $\hat{\tau}$ for τ in $\hat{\boldsymbol{\mu}}_{\boldsymbol{a}}(\tau)$, we obtain the MLE for $\hat{\boldsymbol{\mu}}_{\boldsymbol{a}}$ given $\hat{\tau}$:

$$\hat{\boldsymbol{\mu}}_{\boldsymbol{a}}(\hat{\tau}) = \frac{1}{T-\hat{\tau}} \sum_{t=\hat{\tau}+1}^{T} \boldsymbol{x}_t = \overline{\boldsymbol{x}_{T,\hat{\tau}}} \quad \textbf{(3.19)}$$

where both point estimates, $\hat{\tau}$ and $\hat{\mu}_a$, are maximum likelihood estimators for τ and μ_a.

The MMRC implementation is basically identical to the UMRC. Both charts calculate the maximum *R* value over all possible τ change points. After signaling, the τ maximizing $R_{\max}$ becomes $\hat{\tau}$ and is an input to $\hat{\boldsymbol{\mu}}_{\boldsymbol{a}}$ to generate the MLE for $\hat{\boldsymbol{\mu}}_{\boldsymbol{a}}$ given $\hat{\tau}$. This process is repeated after every chart signal. The main difference between the

MMRC and the UMRC is the added complexity in calculating $\boldsymbol{S}^{-1}$ and the subsequent matrix multiplication in Equation **(3.17)**.

3.4 MMRC Example

Figure 3.1 shows a graphical example of a ten variable MMRC chart with $\tau = 50$ and an observation matrix $\boldsymbol{X}$ pulled from a standard multivariate normal distribution. The mean shift $\boldsymbol{\delta} = [.474, .474, ..., .474]'$ corresponds with a vector norm distance of 1.5. The top chart shows $R_{\max}$ plotted over *t* taking ten observations to signal after $\tau = 50$. The blue line corresponds to the CL, *B* = 14.75, calibrated to $ARL_0 = 200$. For the bottom chart, instead of charting ten individual $\hat{\boldsymbol{\mu}}_a$ vectors, the mean of each $\hat{\boldsymbol{\mu}}_a$ vector is charted over time. This works because the average of $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_a$ is known and equal to 0 and .474 respectfully. As you can see, $\hat{\boldsymbol{\mu}}_a$ hovers around $\boldsymbol{\delta}$ after the second out-of-control observation until the chart signals. However, this chart is for demonstration purposes only because in practice $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_a$ are unknown. Lastly, using the MLE for $\hat{\tau}$ in Function **(3.18)**, we find the estimate is coincidentally equal to the actual known value for τ .

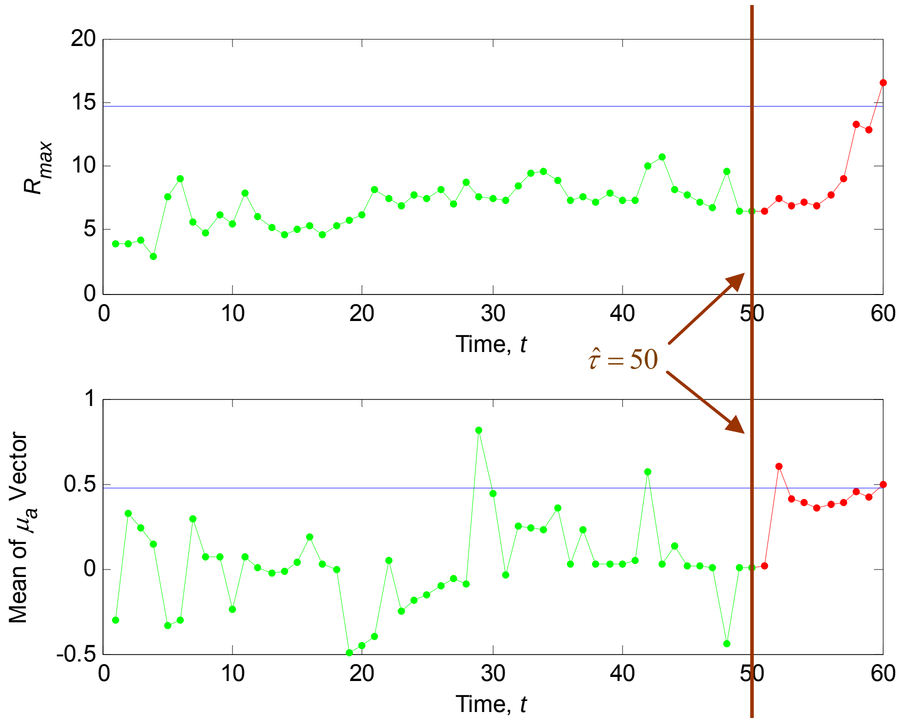

Figure 3.1: Single run MMRC chart with $\tau = 50$, $p = 10$ and mean shift $\boldsymbol{\delta} = [.474, .474, \ldots, .474]'$

3.5 Finding MMRC Control Limits

While the MMRC eliminates the need to tune the control chart for a specific step change magnitude, one still needs a way to find CLs, *B*. To develop a heuristic for estimating *B*, the fact a higher *B* value results in a higher ARL_0 was utilized. The pseudocode in Figure 3.2 shows how the process works. The code starts with a target ARL_0 value with an error of 1% on each side. In the case of Figure 3.2 this equates to a desired error rage for ARL_0 of 303 to 297. Essentially, the heuristic uses a loop to generate candidate ARL_0 values from the MMRC simulation. A range is generated using each candidate ARL_0 value plus/minus the standard error, and then this range matched up to the specified error range.

To match the desired and MMRC simulation error ranges, the heuristic uses four conditions labeled (1), (2), (3) and (4) in Figure 3.2. The program starts with a run size, denoted by N, equal to ten, the CL *B*, denoted by B, equal to one and a *B* increment, denoted as B_shift, equal to one. Condition (3) keeps incrementing B by B_shift until the simulation returns a candidate ARL_0 within the desired upper and lower target range (Condition (1)) or the candidate ARL_0 is within one standard error of the target (Condition (2)). Typically Condition (2) is met first, and B_shift is then divided by three and N is multiplied by ten. For reasons of computational speed, Condition (2) truncates the heuristic if the size of N is greater than 10,000. If B is incremented greater than one standard error from the target, then Condition (4) decrements B by B_shift/3. This asynchronous incrementing and decrementing of B guarantees the same B value is not used twice within each run of the heuristic. The heuristic ends when either the desired range in (Condition (1)) is met or Condition (2) truncates because B is greater than 10,000.

```
target = 300;                       %target value
upper = target + target * .01;      %upper error limit
lower = target + target * .01;      %lower error limit
B=1;                                %control limit
B_shift = 1;                        %amount to change B by
N = 10;                             %number of simulation runs

while done == false
    get ARL, se {standard error} from control chart;

(1) if ARL plus/minus se within upper and lower error bounds then
        %h found, so stop program
        done = true;

(2) else if range of ARL + se AND ARL - se contain target
        %narrow focus
        B_shift = B_shift/3;
        %increase number of simulation runs
        N = N * 10;

        if (N > 10000) then
            %10000 runs yields a small se, so stop for
            %computational and time reasons
            done = true;
        end if
(3) else if ARL + se <= target then
            %increase h
            B = B + B_shift;
(4) else if ARL - se > target then
            %decrease h by a smaller amount
            B = B - B_shift/4;
    end if
end while
```

Figure 3.2: Find *B* Heuristic Psuedocode

In practice, depending on the target ARL_0 and p, the heuristic takes two to eight hours to run on a 2.8 GHz Pentium® 4 processor running Matlab® 7. As one might expect, the higher ARL_0 and p input into the heuristic, the longer it takes to converge.

Figure 3.3 shows a single execution of the find B heuristic with p = 10 and a target $ARL_0 = 300$.

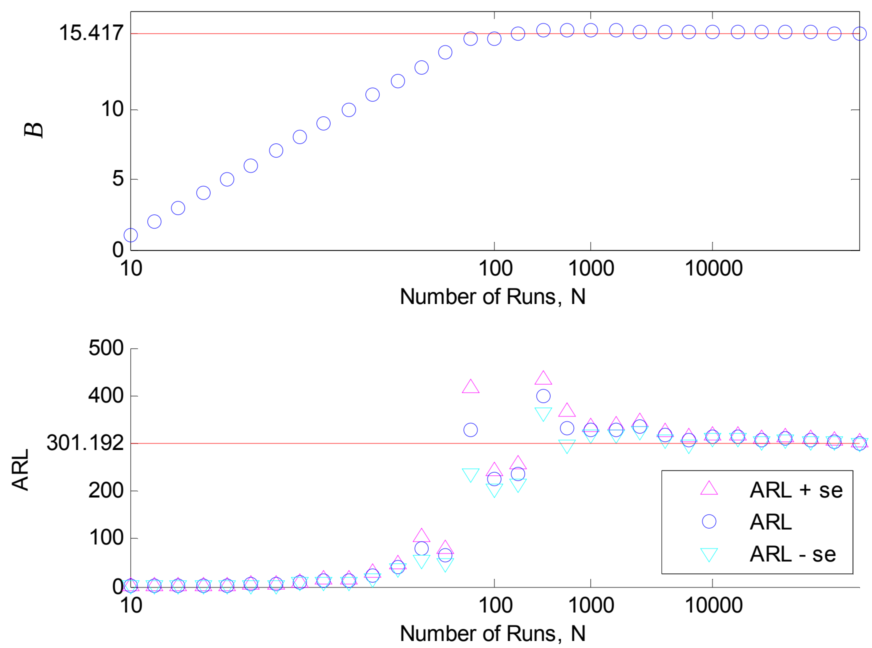

Figure 3.3: Find *B* Heuristic Execution for Target ARL = 300

The top graph shows the change in *B* and the bottom graph shows the change in ARL and standard error from the MMRC simulation. Red lines indicate the final values for *B* and ARL_0. While *B* does not seem to vary much once it reaches around 15, the ARL is highly sensitive (in terms of variability) to the value of *B* and the number of runs. You can see on the bottom graph the effect of over-shooting ARL_0 and then backtracking to the target value. This particular example truncated with a range of 304.099-298.285 versus the desired 303-297 range. Even though this is not the exact range desired, this heuristic produces highly accurate estimates and is more than adequate for the purpose of this thesis.

3.6 False Alarms

For reasons of consistency and fair comparison, this thesis will follow the conventions of Pignatiello and Simpson [12] for the simulation modeling of false alarms. A false alarm occurs when $\tau > 0$ and the chart signals at any time *t*, where $t \le \tau$. If a false alarm occurs, then the control chart is zeroed out and the process is restarted at time $t + 1$. However, the step change will still occur at time τ with $\tau + 1$ as the first out-of-control value. This models the real world where an investigative study finds a true false alarm. In the real world, finding a false alarm indicates the process is truly in-control, the process is then restarted at time $t + 1$ without changing the process inputs. Furthermore, this real world false alarm would not impact the actual but unknown change point.

For example, let $\tau = 50$ and the chart signals at $t = 30$. This signal is considered a false alarm because $t \le \tau$. The control chart is then zeroed out and restarted as if the next observation was the first. However instead of 50 in-control observations, the control chart has only 20 observations until the step change occurs at $\tau = 50$. Thus on the 21st observation, a step change is first recorded on the restarted control chart.

3.7 MC1 and MEWMA Charting Statistics

Since Chapter 4 will compare the MMRC with the MC1 and MEWMA, it is necessary to show and discuss these control charting strategies prior to discussing results.

3.7.1 MC1

The MC1 was developed by Pignatiello and Runger [11]. It is based on the cumulative sum

$$\boldsymbol{C}_t = \sum_{i=t-n_t+1}^{t} \left(\boldsymbol{X}_i - \boldsymbol{\mu_0} \right) \tag{3.20}$$

where $\boldsymbol{X}_i$ is the observation vector at time *i*, $\boldsymbol{\mu_0}$ is the in-control mean vector,

$$n_t = \begin{cases} n_{t-1} + 1 & \text{if } MC1_{t-1} > 0 \\ 1 & \text{if otherwise} \end{cases} \tag{3.21}$$

and the charting statistic is

$$MC1_t = \max\left\{ \sqrt{\boldsymbol{C}_t' \boldsymbol{S}^{-1} \boldsymbol{C}_t} - kn_t, 0 \right\}. \tag{3.22}$$

Here $\boldsymbol{S}$ represents the covariance matrix and *k* is the tuning parameter used to dampen the Mahalanobis distance $\sqrt{\boldsymbol{C}_t' \boldsymbol{S}^{-1} \boldsymbol{C}_t}$ at time *t*. If the MC1 exceeds the CL *h*, then the chart signals. Furthermore, *k* corresponds to one-half of the Euclidean distance the QE wishes to detect.

For example, take the two variable case with $\boldsymbol{X} = \{\boldsymbol{X}_1, \boldsymbol{X}_2\}$. Suppose the expected shift in $\boldsymbol{X}$ is $\{-1.5, .5\}$. The resulting value of *k* is $\left(\sqrt{(-1.5)^2 + (.5)^2} \right) / 2$ = .7906. Like the univariate CUSUM, the MC1 renews itself (*i.e.* zeros out) periodically when the process is in-control. This is controlled in the MC1 by the counter variable n_t in **(3.21)**, and $n_t = 0$ upon startup. For more information on the MC1 see Pignatiello and Runger [11].

3.7.2 MEWMA

The MEWMA developed by Lowry *et al.* [8] is a natural extension of the EMWA. The MEWMA charting statistic is

$$T_t^2 = \boldsymbol{Z}_t' \boldsymbol{S}_{\boldsymbol{Z}_t}^{-1} \boldsymbol{Z}_t \qquad \textbf{(3.23)}$$

where T_t^2 is the MEWMA statistic at the t^{th} observation and $\boldsymbol{S}_{\boldsymbol{Z}_t}^{-1}$ is the weighted covariance matrix defined by either **(3.26)** or **(3.27)**. When T_t^2 exceeds the control limit h_4, the chart signals. Note the subscript of 4 in h_4 has no other significance other than to differentiate h_4 from other CLs using h, such as the MC1. Equation **(3.23)** is a variation of Hotelling's T^2 statistic with the replacement of $\boldsymbol{Z}$ for $\boldsymbol{X}$. $\boldsymbol{Z}_t$ is calculated as

$$\boldsymbol{Z}_t = \boldsymbol{R}\boldsymbol{X}_t + (\boldsymbol{I} - \boldsymbol{R})\boldsymbol{Z}_{t-1}. \qquad \textbf{(3.24)}$$

Here $\boldsymbol{R}$ is a diagonal matrix of weights greater than zero and less than one used to calibrate the chart and $\boldsymbol{I}$ is the identity matrix. The value of $\boldsymbol{Z}_t$ at startup is zero. If $r_1 = r_2 = ... = r_p = r$ in $\boldsymbol{R}$, then **(3.24)** is

$$\boldsymbol{Z}_t = r\boldsymbol{X}_t + (1 - r)\boldsymbol{Z}_{t-1}. \qquad \textbf{(3.25)}$$

The last calculation required is the weighted covariance matrix $\boldsymbol{S}_{\boldsymbol{Z}_t}^{-1}$ of the standard covariance matrix $\boldsymbol{S}$. Assuming the weights are equal, this matrix is calculated one of two ways. Lowry *et al.* [8] found an equation to find the exact or actual covariance matrix based on the t^{th} observation:

$$\boldsymbol{S}_{\boldsymbol{Z}_t} = \left\{ \frac{r\left[1 - \left(1 - r\right)^{2t}\right]}{2 - r} \right\} \boldsymbol{S}. \qquad \textbf{(3.26)}$$

The other method is to assume the chart is fully warmed up and in a steady-state. This steady state covariance matrix is

$$\boldsymbol{S}_{\boldsymbol{Z}_t} = \left\{ \frac{r}{2 - r} \right\} \boldsymbol{S}. \qquad \textbf{(3.27)}$$

Note, in either assumption when $r = 1$, we are left with Hotelling's T^2 chart, which is a special case of the MEWMA. Figure 3.4 illustrates the convergence of the actual weight to the steady state weight. As the graph shows, the two converge on 22nd observation.

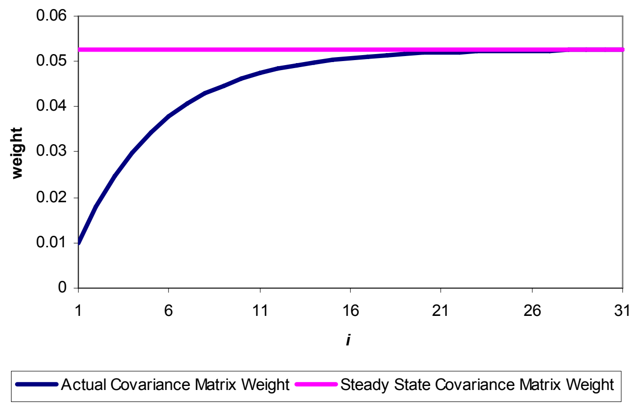

Figure 3.4: Actual vs. Steady State Covariance Matrix Weights

For more information on the MEWMA, consult the paper by Lowry *et al.* [8].

3.8 Relative Mean Index

When conducting an ARL comparison of control charts, some charts have different detection capabilities at different mean shift magnitudes, denoted by δ. For example, consider two control charts, say, Chart 1 and Chart 2. Chart 1 shows superior performance when $\delta \le 2$, but poor performance when $\delta > 2$. Chart 2 performs in an opposite fashion with poor performance when $\delta \le 2$, but superior performance when $\delta > 2$. The question becomes: "Which chart is better?" To answer this question, the RMI developed by Han and Tsung [4] is used. The RMI is a simple weighted average of the compared charts for a desired range of δ.

The calculation of the RMI score is:

$$RMI(c) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{ARL(c)_{\delta_i} - ARL^*_{\delta_i}}{ARL^*_{\delta_i}}\right], \forall \delta_i > 0 \quad \textbf{(3.28)}$$

where c is the desired chart, δ_i is the i^{th} mean shift magnitude, $ARL(c)_{\delta_i}$ is the ARL at chart c and δ_i and $ARL^*_{\delta_i}$ is the smallest ARL over all charts at δ_i. This gives a single value of the chart's overall detection performance, and the lower the RMI score, the better the performance. An RMI of zero indicates the chart has the quickest out-of-control detection of the entire range of tested mean shifts. Note, if an inappropriate range of δ were chosen, say $[0.1, 0.2, 0.3, 0.4, 0.5, 1.0, 2.0]$, then this range of δ would skew the RMI toward charts with superior detection performance for small δ. Overall, when a realistic δ range is chosen, the RMI is an excellent tool when conducting an ARL performance comparison of control charts.

3.9 Conclusion

The MMRC is a much needed addition to the field of SPC. By deriving and then extending the UMRC into the multivariate realm, a magnitude robust control chart for multivariate mean shift detection was developed. Magnitude robust means the QE no longer has to calibrate the chart to a specific mean shift magnitude. This was accomplished using a log-likelihood-ratio and MLE to test for an out-of-control process. Another significant advantage of this method is the MMRC outputs an MLE for the actual change point τ, denoted by $\hat{\tau}$. Use of $\hat{\tau}$ can significantly reduce a QE's search for causality when the chart signals. This results in less down time and allows for greater productivity for the process. Furthermore, the MMRC is identical to the UMRC when $p = 1$, and therefore can supplant the UMRC.

In addition, a simulation-based search heuristic was presented to find the control limit *B* for the MMRC. This heuristic allows a QE to quickly (within a few hours) obtain an estimate for *B*. Next, the handling of false alarms within a simulation was discussed. Essentially, once a false alarm is detected the control chart resets itself, but the change point is not altered. This models the real world where a false alarm is independent of the actual change point.

Finally, this chapter presented the competing charts (MC1 and MEWMA) and the RMI, a method to compare ARL performance of these charts with the MMRC. The MC1 and MEWMA represent the current state of the art in multivariate control charts. As a result, a successful ARL performance comparison of the MMRC with the MC1 and MEWMA is critical to its acceptance in the real world. For these comparisons, the RMI was used to measure control chart performance.

IV. Results

4.1 Introduction

In Chapter III, a control chart using a change point model and likelihood-ratios was derived. In this chapter, the results of this chart, the MMRC, are compared to the MC1 and MEWMA. Note Hotelling's T^2 is not directly considered because it is a special case of the MEWMA. This comparison was accomplished through ARL evaluations over several tuning parameters and two change points.

To evaluate the ARL performance of the MMRC, MC1 and MEWMA, a Monte Carlo simulation is used. This simulation will allow the QE to specify the actual change point, control limit, in-control mean vector, out-of-control mean vector, covariance matrix, tuning parameter (MC1 and MEWMA) and the run size or number of simulated runs to collect. All of these inputs give the QE great flexibility to evaluate the MMRC, MC1, and MEWMA. For outputs, the simulation gives the overall average ARL and standard error, and for the MMRC, it gives the average estimate for the change point.

Although there is an infinite combination of input parameters for the simulation, this research will focus only on a select few to answer three questions. These three questions all refer to the comparison of the ARL performance among the MMRC, MC1, and MEWMA. First off, "What effect does the change point position have?" To answer this question, the results will be split between those where the process is simulated as out-of-control from the beginning and simulated as in-control for 50 observations and then suddenly shifting out-of-control. Within these results the second and third questions are asked: "What effect does the number of out-of-control variables in the out-of-control

vector have?" and "What effect does the number of variables in the process have?" With regard to the number of out-of-control variables question, two situations will be considered: one where the one variable in the out-of-control mean vector shifts and another where all variables in the out-of-control mean shift. Finally, three different variable sizes will answer the last question.

In addition to the ARL performance comparison, a regression will be used to estimate values of *B* in the MMRC. These values will allow for an ARL_0 ranging from 50 to 300 with up to 10 variables.

Next, the robustness of the MMRC change point estimator is evaluated. These estimators will be pulled from each of the ARL performance simulations and compared to the known simulated change point.

Finally, preliminary results from this research effort have shown the MEWMA has different in-control ARL values when the process is out-of-control from the beginning versus when the process is initially in-control and then suddenly shifts out-of-control some time later. Since this phenomenon could adversely effect the ARL performance evaluation, this chapter will research it in depth.

4.2 ARL Performance Simulation Implementation

Although control charts are programmable on many different languages and statistical packages, Matlab® 7 was selected for its rapid coding capabilities and wide variety of built-in statistical functions. However, in this section, pseudocode is used instead of Matlab® code in order to ease understanding. The great advantage to programming the simulation model used for evaluating ARL performance of control

charts is the actual code is very similar from chart to chart. The only real difference is calculating the charting statistic. As a result, the same code is slightly modified to calculate ARL performance results for the MMRC, MC1 and MEWMA. Moreover, since the simulation code is quite long, we will break it up into three parts: simulation inputs, main simulation loop, and the single run simulation of the chart itself.

4.2.1 Simulation Inputs

Figure 4.1 shows the standard simulation inputs for a multivariate control chart. The main difference between the charts is Hotelling's T^2 and MMRC do not need the tuning parameter `k`. Variables `tau` and `delta` are the induced and therefore they are the known change point and step change from `mu0`. If you are using standardized data, `mu0` is column vector of zeros from where the number of variables, *p*, is equal to `mu0`'s row dimension.

```
% B = upper control limit
% tau = change point
% N = number of times to run the simulation
% mu0 = in-control average vector
% delta = step change vector
% sigma = covariance matrix
% k = tuning parameter (for non-MMRC charts)
```

Figure 4.1: Simulation Inputs

4.2.2 Main Simulation Loop

The loop in Figure 4.2 is responsible for running the simulation *N* times where *N* is the run size of the simulation. As one can see, there are four internal variables here and three output variables. At the start of each run the chart has not signaled, thus `chart_signals` is false before going into the single run simulation. The other two input

variables are used to handle false alarms. The variable `t_const` advances monitors the time when the actual shift occurs while `t` is reset to zero when false alarms occur. All three of these are fed into the single run simulation, which outputs the sum of ARL, the sum of ARL^2, and the estimate for the change point, `tau_hat`. These values are used to calculate the average ARL, standard error, and `tau_hat`. Note `tau_hat` is only used with the MMRC and not the MC1 or MEWMA.

```
for i = 1 to N    %outer simulation loop
    chart_signals = false; %boolean test for loop termination
    t = 0;              %counter reset after false alarms
    t_const = 0;        %counter not reset after false alarms

    get sum of ARL, squared ARL and tau_hat from the single run _
        simulation;
end

calculate and output average ARL
calculate and output standard error
calculate and output average tau_hat
```

Figure 4.2: Main Simulation Loop

4.2.3 Single Run Simulation

The third component of the simulation is the single run simulation. This section in Figure 4.3 uses a while loop to run until a non-false alarm is detected. To start, both `t` and `t_const` are incremented to the first observations. The `if … then` structure determines whether the p by `t` matrix `X` receives an in or out-of-control random vector of observations. This set of observations is sent to a control chart function, such as R_{max} in **(3.17)**, which outputs the charting statistic, `chart_stat`, and for the MMRC, the potential change point estimate, `cp`. The next couple of `if` statements determine the false alarm status when the chart signals. If a false alarm occurs, then `X` and `t` are reset to zero and the entire chart is restarted. Otherwise the chart truly signals, `chart_signals` is set to

true and `cp` becomes the actual change point estimate, `tau_hat`. See Section 3.6 for more information on false alarms. At the bottom, we tally the ARL and ARL^2 used in Figure 4.2. This information is then sent back to the main simulation loop in Section 4.2.2.

```
while chart_signals == false
        t = t + 1; t_const = t_const + 1;

        if t_const <= tau then  %if process in-control
            X(row of obs.,col t) = multivariate normal with mu0;
        else                    %if process out-of-control
            X(row of obs.,col t) = multivariate normal with mu0 _
                                                     + delta;
        end

        get chart_stat, cp from the appropriate function;

        if (chart_stat > B) and (t_const <= tau) %false alarm occured
            zero out X;
            t = 0;
        end

        if (t_stat > B) and (t_const > tau) %chart signals
            chart_signals = true;
            tau_hat = cp;
        end
    end

    sum the ARL;
    sum the ARL^2;

end
```

Figure 4.3: Single Run Simulation

4.3 ARL Performance ($\tau = 0$)

In their papers, both Pignatiello and Runger [11] and Lowry and Montgomery [7] provide CL values for the MC1 and MEWMA respectively. These values are recomputed using the method in section 3.5 as both an error check and to compensate for different pseudorandom number generators. The proceeding ARL performance evaluation will use these recomputed CL values.

Assuming the process was out-of-control from the beginning, an evaluation of the ARL performance was conducted using Monte Carlo simulation to compare the ARL performances of the MMRC with those of the MC1 and MEWMA. The run size for each ARL simulation was 10,000. Additionally, the covariance matrix was assumed equivalent to the identity matrix (*i.e.* process variables assumed to be uncorrelated).

This study will start with an RMI summary across all tuning parameter settings contained in Tables 4.2 through 4.7. After this RMI comparison, the simulated ARL values are presented with their corresponding standard errors. The ARL performance is evaluated when $p = \{2,3,10\}$. Furthermore, two different types of mean shifts are considered: one where a single variable in the process suddenly shifts $\left(\boldsymbol{\mu}_a = [\delta, 0, 0, ..., 0]\right)$ and another where all variables in the process suddenly shift simultaneously with identical magnitude $\left(\boldsymbol{\mu}_a = [\delta, \delta, \delta, ..., \delta]\right)$.

4.3.1 RMI Comparison

Table 4.1 gives a summary of the RMI scores contained in Tables B.1 through B.6 from Appendix B. The RMI compares the ARL performance of the MMRC, MC1, and MEWMA (both using the actual covariance matrix and the steady state covariance matrix) across the range of tested change magnitudes. These change magnitudes are the same as those in the D_e columns from TablesB.1 through B.6 (see Section 4.3.2 for an explanation). The top row of Table 4.1 gives the chart type, the tuning parameter (if needed), the RMI values when $p = \{2,3,10\}$ for $\boldsymbol{\mu}_a = [\delta, 0, 0, ..., 0]$ and the RMI values when $p = \{2,3,10\}$ for $\boldsymbol{\mu}_a = [\delta, \delta, \delta, ..., \delta]$.

Here lower RMI values equate to better ARL performance and higher RMI values equate to lesser ARL performance over the range of tested change magnitudes. An RMI of zero indicates this chart and tuning parameter was superior over all change points for a given number of variables *p* and the particular $\boldsymbol{\mu}_a$ shift.

Table 4.1: RMI Summary for $\tau = 0$ from Tables B.1 through B.6

Type	Tuning Parameter	$\mu_a = (\delta,0,0,\ldots,0)$			$\mu_a = (\delta,\delta,\delta,\ldots,\delta)$		
		p = 2	*p* = 3	*p* = 10	*p* = 2	*p* = 3	*p* = 10
MMRC	--	0.40	0.40	0.43	0.41	0.41	0.43
MC1	k = 0.25	0.97	1.06	1.42	0.98	1.06	1.43
	k = 0.50	0.61	0.64	0.77	0.61	0.64	0.76
	k = 1.00	0.49	0.51	0.55	0.50	0.51	0.55
MEWMA Actual Covariance Matrix	r = 0.05	0.00	0.00	0.01	0.00	0.00	0.01
	r = 0.1	0.09	0.09	0.12	0.09	0.09	0.12
	r = 0.15	0.15	0.15	0.19	0.15	0.16	0.19
	r = 0.5	0.54	0.59	0.80	0.55	0.58	0.81
	r = 0.8	1.10	1.22	1.76	1.12	1.22	1.74
MEWMA Steady-State Covariance Matrix	r = 0.05	1.14	1.23	1.41	1.14	1.23	1.40
	r = 0.1	0.84	0.87	0.97	0.85	0.87	0.97
	r = 0.15	0.72	0.75	0.80	0.73	0.75	0.80
	r = 0.5	0.64	0.71	0.96	0.65	0.70	0.95
	r = 0.8	1.11	1.26	1.83	1.10	1.26	1.81

Clearly, the MEWMA using the actual covariance matrix and a tuning parameter of *r* = 0.05 is superior to all other charts due to its near zero RMI score. Moreover, all RMI scores under the MEWMA steady-state covariance matrix calculation are inferior to the MEWMA actual covariance matrix, MC1, and MMRC RMI scores. Likewise, the values of *r* at 0.5 and 0.8 also have relatively poor performance to the MC1 and MMRC. As a result, Section 4.3.2 will eliminate these data points in the interest of brevity and defer the complete tables to Appendix B. However, the RMI values are not recomputed for these abbreviated tables to maintain consistency with the Appendix B tables.

4.3.2 Table Comparison

In this section, Tables 4.2 through 4.4 assume $\boldsymbol{\mu}_a = [\delta, 0, 0, ..., 0]$, and Tables 4.5 through 4.7 assume $\boldsymbol{\mu}_a = [\delta, \delta, \delta, ..., \delta]$. Although there exists an infinite number of ways

to define $\boldsymbol{\mu}_a$, using extremes makes interpolation easier and gives a good look at each chart's performance.

The tables are arranged with the estimated ARL on top and the corresponding standard error in parentheses along the bottom. The header row is organized left to right with D_e as the Euclidean distance from the centroid (Equation**(1.1)**), δ as the individual mean shift contained within $\boldsymbol{\mu}_a$, the MMRC column, the MC1 CUSUM columns with tuning parameter settings $k = \{0.25, 0.50, 1.00\}$ and the MEWMA using the actual covariance matrix (Equation **(3.26)**) with tuning parameter settings $r = \{0.05, 0.10, 0.15\}$. The second to the bottom row contains the CL used, and the bottom row presents the RMI values.

Table 4.2: τ = 0, p = 2, μ_a = (δ,0)

			MC1 CUSUM			MEWMA Actual Covariance Matrix		
Distance D_e	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15
0.00	0.00	200.38	199.38	201.14	199.13	200.74	201.92	199.15
		(1.92)	(1.88)	(2.05)	(2.02)	(2.21)	(2.08)	(1.96)
0.25	0.25	87.20	66.86	93.06	128.14	58.93	73.78	85.22
		(0.69)	(0.54)	(0.87)	(1.28)	(0.56)	(0.71)	(0.82)
0.50	0.50	34.15	25.38	31.23	56.27	20.99	25.17	29.10
		(0.22)	(0.15)	(0.26)	(0.55)	(0.17)	(0.20)	(0.25)
0.75	0.75	17.89	14.79	15.08	24.96	10.78	12.66	13.96
		(0.11)	(0.07)	(0.10)	(0.23)	(0.08)	(0.09)	(0.11)
1.00	1.00	11.09	10.31	9.26	12.90	6.82	7.79	8.53
		(0.06)	(0.04)	(0.05)	(0.11)	(0.05)	(0.05)	(0.06)
1.25	1.25	7.63	7.98	6.74	7.79	4.77	5.41	5.71
		(0.04)	(0.03)	(0.03)	(0.06)	(0.03)	(0.03)	(0.04)
1.50	1.50	5.67	6.51	5.20	5.31	3.63	4.04	4.30
		(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)	(0.03)
1.75	1.75	4.44	5.49	4.34	3.97	2.90	3.21	3.35
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
2.00	2.00	3.62	4.78	3.69	3.21	2.36	2.62	2.73
		(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)
2.25	2.25	3.00	4.27	3.25	2.72	2.00	2.19	2.28
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
2.50	2.50	2.55	3.84	2.90	2.32	1.75	1.88	1.98
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
2.75	2.75	2.21	3.52	2.64	2.08	1.55	1.67	1.73
3.00	3.00	1.96	3.25	2.42	1.87	1.40	1.49	1.55
3.25	3.25	1.74	3.02	2.26	1.69	1.29	1.37	1.40
3.50	3.50	1.58	2.82	2.12	1.56	1.20	1.26	1.30
3.75	3.75	1.45	2.64	2.01	1.44	1.13	1.18	1.21
4.00	4.00	1.32	2.48	1.93	1.34	1.09	1.12	1.14
4.25	4.25	1.24	2.34	1.83	1.25	1.05	1.08	1.09
4.50	4.50	1.17	2.22	1.76	1.18	1.03	1.05	1.06
4.75	4.75	1.11	2.13	1.68	1.12	1.02	1.02	1.03
5.00	5.00	1.07	2.06	1.58	1.08	1.01	1.02	1.02
		B = 6.66	h = 7.52	h = 4.78	h = 2.69	h4 = 7.71	h4 = 8.79	h4 = 9.36
RMI:		0.40	0.97	0.61	0.49	0.00	0.09	0.15

In Table 4.2, the MEWMA with r = .05 is superior to both the MMRC and MC1 for every magnitude of change tested. The distinction between the MMRC and MC1 is less clear. As typical with most CUSUM based charts, the MC1 has difficulty detecting large shifts unless it is tuned with a large k value, which, in turn, increase the number of observations to detect small shifts. As expected, an exactly tuned MC1 at $k = \{0.25, 0.50, 1.00\}$ corresponding to $\delta = \{0.50, 1.00, 2.00\}$ readily outperforms the MMRC. However, considering the entire range of possible shifts, the MMRC is superior to the MC1 over all three tuning parameters with an RMI = 0.40.

Table 4.3: τ = 0, p = 3, μ_a = (δ,0,0)

			MC1 CUSUM			MEWMA Actual Covariance Matrix		
Distance D_e	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15
0.00	0.00	201.01	200.34	201.59	199.93	202.34	202.68	199.02
		(1.96)	(1.89)	(1.99)	(1.98)	(2.17)	(2.04)	(2.00)
0.25	0.25	95.05	71.77	99.77	139.77	67.33	84.09	94.74
		(0.74)	(0.59)	(0.96)	(1.37)	(0.63)	(0.81)	(0.92)
0.50	0.50	37.43	27.86	34.41	63.91	23.53	28.77	34.02
		(0.24)	(0.17)	(0.29)	(0.61)	(0.19)	(0.24)	(0.29)
0.75	0.75	19.72	16.16	16.46	28.46	12.38	14.43	16.00
		(0.12)	(0.07)	(0.11)	(0.26)	(0.09)	(0.10)	(0.12)
1.00	1.00	12.24	11.39	10.16	14.29	7.67	8.75	9.57
		(0.07)	(0.04)	(0.05)	(0.12)	(0.05)	(0.06)	(0.06)
1.25	1.25	8.46	8.82	7.30	8.47	5.38	6.04	6.48
		(0.04)	(0.03)	(0.03)	(0.06)	(0.03)	(0.04)	(0.04)
1.50	1.50	6.27	7.27	5.77	5.80	3.99	4.46	4.76
		(0.03)	(0.02)	(0.02)	(0.04)	(0.02)	(0.03)	(0.03)
1.75	1.75	4.88	6.15	4.73	4.30	3.18	3.56	3.70
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
2.00	2.00	3.93	5.39	4.05	3.48	2.59	2.86	3.02
		(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.02)
2.25	2.25	3.29	4.81	3.57	2.91	2.19	2.41	2.52
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
2.50	2.50	2.78	4.32	3.20	2.54	1.93	2.07	2.16
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
2.75	2.75	2.42	3.94	2.89	2.24	1.67	1.82	1.89
3.00	3.00	2.12	3.64	2.65	2.03	1.50	1.62	1.68
3.25	3.25	1.88	3.38	2.46	1.86	1.37	1.47	1.52
3.50	3.50	1.71	3.17	2.30	1.71	1.27	1.34	1.38
3.75	3.75	1.55	2.99	2.18	1.59	1.19	1.24	1.27
4.00	4.00	1.43	2.83	2.08	1.48	1.13	1.17	1.20
4.25	4.25	1.32	2.68	2.00	1.37	1.08	1.12	1.14
4.50	4.50	1.23	2.53	1.94	1.29	1.05	1.07	1.09
4.75	4.75	1.16	2.38	1.88	1.21	1.03	1.04	1.06
5.00	5.00	1.11	2.26	1.82	1.14	1.02	1.03	1.03
		B = 7.94	h = 8.79	h = 5.55	h = 3.15	h4 = 9.82	h4 = 10.99	h4 = 11.57
RMI:		0.40	1.06	0.64	0.51	0.00	0.09	0.15

As p increases, the number of samples required to detect a one variable shift increases, especially under a smaller magnitude shift. Otherwise, the same conclusion for Table 4.2 holds for Table 4.3.

Table 4.4: τ = 0, p = 10, μ_a = (δ,0,0,…,0)

			MC1 CUSUM			MEWMA Actual Covariance Matrix		
Distance D_e	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15
0.00	0.00	200.13	200.58	201.14	198.96	205.21	204.17	198.00
		(1.92)	(2.06)	(2.05)	(1.94)	(2.16)	(2.08)	(2.02)
0.25	0.25	118.79	84.20	122.42	158.52	93.34	118.13	128.21
		(0.94)	(0.78)	(1.19)	(1.62)	(0.92)	(1.18)	(1.28)
0.50	0.50	51.71	33.47	44.65	88.47	34.85	44.94	54.58
		(0.33)	(0.17)	(0.41)	(0.89)	(0.28)	(0.39)	(0.50)
0.75	0.75	27.66	21.56	19.69	40.30	17.78	21.62	25.60
		(0.15)	(0.08)	(0.13)	(0.39)	(0.13)	(0.16)	(0.21)
1.00	1.00	17.16	16.09	12.64	19.32	11.02	12.74	14.38
		(0.09)	(0.05)	(0.06)	(0.17)	(0.07)	(0.08)	(0.10)
1.25	1.25	11.85	12.86	9.50	10.67	7.73	8.69	9.41
		(0.06)	(0.03)	(0.03)	(0.08)	(0.05)	(0.05)	(0.06)
1.50	1.50	8.68	10.74	7.74	7.00	5.70	6.39	6.84
		(0.04)	(0.02)	(0.02)	(0.04)	(0.03)	(0.04)	(0.04)
1.75	1.75	6.74	9.32	6.53	5.30	4.43	4.96	5.20
		(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.03)	(0.03)
2.00	2.00	5.40	8.19	5.69	4.37	3.60	3.94	4.17
		(0.02)	(0.02)	(0.01)	(0.02)	(0.02)	(0.02)	(0.02)
2.25	2.25	4.46	7.31	5.07	3.69	3.00	3.28	3.44
		(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)
2.50	2.50	3.77	6.66	4.54	3.29	2.58	2.81	2.90
		(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
2.75	2.75	3.23	6.10	4.17	2.95	2.23	2.41	2.54
3.00	3.00	2.83	5.61	3.83	2.69	1.99	2.14	2.22
3.25	3.25	2.50	5.22	3.57	2.47	1.78	1.89	1.98
3.50	3.50	2.22	4.87	3.35	2.30	1.61	1.71	1.79
3.75	3.75	2.01	4.58	3.15	2.18	1.47	1.57	1.62
4.00	4.00	1.82	4.33	2.99	2.08	1.35	1.45	1.48
4.25	4.25	1.67	4.11	2.84	2.00	1.27	1.34	1.38
4.50	4.50	1.55	3.92	2.70	1.92	1.21	1.26	1.29
4.75	4.75	1.44	3.74	2.56	1.85	1.14	1.19	1.21
5.00	5.00	1.34	3.56	2.42	1.78	1.09	1.14	1.15
		B = 14.75	h = 15.33	h = 9.58	h = 5.53	h4 = 21.41	h4 = 22.98	h4 = 23.7
RMI:		0.43	1.42	0.77	0.55	0.01	0.12	0.19

Again, as p increases to ten, the number of samples required to detect a one variable shift increases. However, this sample increase appears nonlinear and decreases as p increases for all charts and tuning parameters. For example the MMRC at $\delta = .25$ goes from 87.2 (p = 2) to 95.05 (p = 3) and then to 118.79 (p = 10). This is a positive result if one has many variables within their process.

Table 4.5: τ = 0, p = 2, μ_a = (δ,δ)

			MC1 CUSUM			MEWMA Actual Covariance Matrix		
Distance D_e	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15
0.00	0.00	200.38	199.38	201.14	199.13	202.45	196.10	200.42
		(1.92)	(1.88)	(2.05)	(2.02)	(2.17)	(2.01)	(2.02)
0.25	0.18	87.99	67.42	92.54	129.03	58.86	73.48	84.35
		(0.68)	(0.55)	(0.87)	(1.28)	(0.56)	(0.70)	(0.83)
0.50	0.35	33.85	25.44	31.00	56.59	20.44	25.00	28.91
		(0.22)	(0.15)	(0.25)	(0.55)	(0.16)	(0.21)	(0.25)
0.75	0.53	17.65	14.74	15.11	25.11	10.90	12.78	13.85
		(0.11)	(0.07)	(0.10)	(0.23)	(0.08)	(0.09)	(0.10)
1.00	0.71	11.18	10.31	9.23	12.89	6.84	7.78	8.46
		(0.06)	(0.04)	(0.05)	(0.11)	(0.05)	(0.05)	(0.06)
1.25	0.88	7.64	7.97	6.76	7.75	4.83	5.41	5.77
		(0.04)	(0.03)	(0.03)	(0.06)	(0.03)	(0.03)	(0.04)
1.50	1.06	5.68	6.51	5.23	5.32	3.63	4.00	4.28
		(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)	(0.02)
1.75	1.24	4.41	5.50	4.35	3.98	2.87	3.18	3.36
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
2.00	1.41	3.59	4.80	3.73	3.20	2.34	2.62	2.74
		(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)
2.25	1.59	2.99	4.27	3.25	2.72	1.98	2.21	2.28
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
2.50	1.77	2.56	3.84	2.90	2.33	1.74	1.89	1.97
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
2.75	1.94	2.21	3.52	2.64	2.08	1.54	1.67	1.72
3.00	2.12	1.95	3.25	2.43	1.88	1.40	1.48	1.55
3.25	2.30	1.75	3.02	2.26	1.70	1.27	1.36	1.41
3.50	2.47	1.57	2.82	2.13	1.56	1.20	1.27	1.30
3.75	2.65	1.45	2.64	2.02	1.44	1.13	1.18	1.21
4.00	2.83	1.33	2.49	1.92	1.34	1.09	1.12	1.14
4.25	3.01	1.24	2.33	1.84	1.25	1.05	1.08	1.09
4.50	3.18	1.17	2.22	1.76	1.18	1.03	1.05	1.06
4.75	3.36	1.11	2.13	1.67	1.12	1.02	1.03	1.03
5.00	3.54	1.08	2.07	1.57	1.08	1.01	1.02	1.02
		B = 6.66	h = 7.52	h = 4.78	h = 2.69	h4 = 7.71	h4 = 8.79	h4 = 9.36
RMI:		0.41	0.98	0.61	0.50	0.00	0.09	0.15

Here in Table 4.5, all process variable means have shifted instead of simply one. In order to maintain the same Euclidean distance from the centroid, each δ is calculated as follows:

$$D_e = \sqrt{p\,\delta^2} \Rightarrow \delta = \sqrt{\frac{D_e^2}{p}}$$

where δ is the mean shift contained in $\boldsymbol{\mu}_a$. For example, with p = 2 at a distance D_e = .5 results in a total shift of $\delta = \sqrt{\frac{0.5^2}{2}} = .35$. Interestingly, this does not seem to have an impact on the ARL detection capabilities when compared to the single shift $\boldsymbol{\mu}_a$ because

the Euclidean distances are the same. In fact, this table is almost indistinguishable from Table 4.2.

Table 4.6: τ = 0, p = 3, μ_a = (δ,δ,δ)

			MC1 CUSUM			MEWMA Actual Covariance Matrix		
Distance D_e	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15
0.00	0.00	201.01	200.34	201.59	199.93	203.24	201.85	203.96
		(1.96)	(1.89)	(1.99)	(1.98)	(2.18)	(2.09)	(2.10)
0.25	0.14	96.21	73.63	99.94	141.63	66.91	82.28	97.73
		(0.75)	(0.61)	(0.95)	(1.42)	(0.62)	(0.78)	(0.97)
0.50	0.29	37.94	27.88	34.60	63.66	23.47	28.97	34.30
		(0.25)	(0.17)	(0.29)	(0.61)	(0.19)	(0.24)	(0.30)
0.75	0.43	19.76	16.20	16.42	28.15	12.37	14.30	16.37
		(0.12)	(0.07)	(0.11)	(0.26)	(0.09)	(0.10)	(0.12)
1.00	0.58	12.26	11.45	10.14	14.42	7.70	8.70	9.59
		(0.07)	(0.04)	(0.05)	(0.12)	(0.05)	(0.06)	(0.06)
1.25	0.72	8.45	8.84	7.33	8.55	5.40	6.01	6.41
		(0.04)	(0.03)	(0.03)	(0.06)	(0.03)	(0.04)	(0.04)
1.50	0.87	6.27	7.27	5.75	5.72	4.02	4.47	4.77
		(0.03)	(0.02)	(0.02)	(0.04)	(0.02)	(0.03)	(0.03)
1.75	1.01	4.91	6.16	4.72	4.33	3.18	3.56	3.69
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
2.00	1.15	3.92	5.36	4.06	3.47	2.59	2.91	3.00
		(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.02)	(0.02)
2.25	1.30	3.32	4.79	3.57	2.92	2.21	2.41	2.51
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
2.50	1.44	2.78	4.32	3.20	2.53	1.90	2.06	2.16
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
2.75	1.59	2.43	3.95	2.90	2.24	1.69	1.83	1.87
3.00	1.73	2.12	3.63	2.66	2.02	1.49	1.62	1.68
3.25	1.88	1.89	3.38	2.46	1.85	1.37	1.47	1.52
3.50	2.02	1.69	3.17	2.31	1.71	1.27	1.33	1.38
3.75	2.17	1.55	2.99	2.18	1.58	1.19	1.25	1.28
4.00	2.31	1.42	2.83	2.08	1.47	1.13	1.18	1.20
4.25	2.45	1.32	2.68	2.00	1.37	1.08	1.11	1.14
4.50	2.60	1.23	2.53	1.95	1.29	1.05	1.08	1.08
4.75	2.74	1.16	2.38	1.88	1.21	1.03	1.05	1.06
5.00	2.89	1.11	2.27	1.82	1.14	1.02	1.03	1.03
		B = 7.94	h = 8.79	h = 5.55	h = 3.15	h4 = 9.82	h4 = 10.99	h4 = 11.57
RMI:		0.41	1.06	0.64	0.51	0.00	0.09	0.16

Likewise, Table 4.6 and Table 4.7 require more samples to detect when additional variables are added, and have a similar performance to Table 4.3 and Table 4.4.

Table 4.7: τ = 0, p = 10, μ_a = (δ,δ,δ,…,δ)

			MC1 CUSUM			MEWMA Actual Covariance Matrix		
Distance D_e	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15
0.00	0.00	200.13	200.58	201.14	198.96	203.74	202.87	200.72
		(1.92)	(2.06)	(2.05)	(1.94)	(2.19)	(2.07)	(2.03)
0.25	0.08	119.83	85.63	120.92	160.33	92.45	116.94	129.88
		(0.94)	(0.81)	(1.20)	(1.61)	(0.91)	(1.17)	(1.29)
0.50	0.16	51.82	33.65	44.42	88.95	34.44	44.51	55.56
		(0.33)	(0.17)	(0.41)	(0.89)	(0.28)	(0.38)	(0.50)
0.75	0.24	27.54	21.60	19.81	41.49	18.01	21.59	25.91
		(0.16)	(0.08)	(0.13)	(0.41)	(0.13)	(0.16)	(0.21)
1.00	0.32	17.25	16.05	12.67	19.19	11.11	12.89	14.37
		(0.09)	(0.05)	(0.06)	(0.18)	(0.07)	(0.08)	(0.10)
1.25	0.40	11.92	12.93	9.51	10.55	7.71	8.70	9.45
		(0.06)	(0.03)	(0.03)	(0.08)	(0.05)	(0.05)	(0.06)
1.50	0.47	8.73	10.79	7.72	7.06	5.66	6.38	6.72
		(0.04)	(0.02)	(0.02)	(0.04)	(0.03)	(0.04)	(0.04)
1.75	0.55	6.73	9.28	6.56	5.35	4.41	4.96	5.17
		(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.03)	(0.03)
2.00	0.63	5.38	8.18	5.68	4.38	3.59	4.02	4.19
		(0.02)	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)	(0.02)
2.25	0.71	4.44	7.34	5.05	3.73	3.00	3.30	3.46
		(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)
2.50	0.79	3.75	6.66	4.56	3.29	2.58	2.81	2.91
		(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
2.75	0.87	3.24	6.08	4.15	2.93	2.24	2.44	2.52
3.00	0.95	2.83	5.61	3.84	2.67	1.97	2.15	2.23
3.25	1.03	2.48	5.23	3.56	2.47	1.77	1.90	1.96
3.50	1.11	2.23	4.88	3.33	2.31	1.60	1.72	1.79
3.75	1.19	2.01	4.59	3.16	2.18	1.47	1.58	1.62
4.00	1.26	1.83	4.33	2.99	2.08	1.36	1.44	1.48
4.25	1.34	1.68	4.11	2.84	1.99	1.27	1.34	1.38
4.50	1.42	1.54	3.92	2.69	1.91	1.21	1.26	1.27
4.75	1.50	1.44	3.75	2.56	1.85	1.14	1.18	1.21
5.00	1.58	1.34	3.55	2.41	1.78	1.09	1.13	1.16
		B = 14.75	h = 15.33	h = 9.58	h = 5.53	h4 = 21.41	h4 = 22.98	h4 = 23.7
RMI:		0.43	1.43	0.76	0.55	0.01	0.12	0.19

4.4 ARL Performance (τ = 50)

At $\tau = 50$, the MEWMA is no longer the superior chart and is outperformed by the MMRC. Furthermore, the same underlying assumptions from Section 4.3 hold in this section with the exception of $\tau = 50$ instead of $\tau = 0$.

4.4.1 RMI Summary

Table 4.8 is another RMI summary similar to Table 4.1 from Section 4.3.1 except $\tau = 50$ and the scores are from Tables B.7 through B.12. Moreover, the same assumptions from Section 4.3.1 apply here as well.

Table 4.8: RMI Summary for $\tau = 50$ from Tables B.7 through B.12

Type	Tuning Parameter	$\mu_a = (\delta,0,0,\ldots,0)$ $p = 2$	$p = 3$	$p = 10$	$\mu_a = (\delta,\delta,\delta,\ldots,\delta)$ $p = 2$	$p = 3$	$p = 10$
MMRC	--	0.07	0.08	0.09	0.07	0.07	0.09
MC1	k = 0.25	0.76	0.92	1.43	0.76	0.91	1.43
	k = 0.50	0.42	0.51	0.91	0.42	0.51	0.91
	k = 1.00	0.30	0.37	0.61	0.30	0.37	0.61
MEWMA Actual Covariance Matrix	r = 0.05	0.55	0.53	0.39	0.56	0.52	0.39
	r = 0.1	0.39	0.39	0.28	0.39	0.38	0.28
	r = 0.15	0.31	0.31	0.27	0.30	0.31	0.26
	r = 0.5	0.31	0.35	0.54	0.31	0.35	0.54
	r = 0.8	0.66	0.75	1.28	0.66	0.75	1.28
MEWMA Steady-State Covariance Matrix	r = 0.05	0.65	0.65	0.58	0.65	0.65	0.58
	r = 0.1	0.42	0.42	0.38	0.42	0.42	0.38
	r = 0.15	0.33	0.33	0.30	0.33	0.33	0.30
	r = 0.5	0.29	0.35	0.55	0.29	0.35	0.55
	r = 0.8	0.65	0.79	1.30	0.65	0.78	1.29

Looking at the MEWMA actual covariance matrix with $r = 0.05$, we see this chart and tuning parameter is no longer superior in terms of the RMI score. In fact, among all of the MEWMA and MC1 tuning parameters, the MEWMA with $r = 0.15$ under the actual covariance matrix has a lower RMI score for $p = \{3,10\}$. Only the MEWMA steady state covariance matrix with $r = 0.5$ and the MC1 with $k = 1.00$ have a lower RMI score than the actual covariance matrix MEWMA with $r = 0.15$ when $p = 2$. Even then the RMI scores only have a maximum difference of 0.02 between them.

Regardless, the MMRC possesses the lowest RMI score with values less than or equal to 0.09 for all $p = \{2,3,10\}$ and both configurations of $\boldsymbol{\mu}_a$. The stark contrast of the MMRC RMI scores versus the MC1 and MEWMA RMI scores is due to the narrow range of potential mean shift magnitudes where the MC1 and MEWMA have quick detection and the fact the chart is in-control for 50 observations. This is shown in Tables B.7 through B.12 and discussed in Section 4.4.2.

4.4.2 Table Comparison

In this section, Tables 4.9 through 4.11 assume $\boldsymbol{\mu}_a = [\delta, 0, 0, ..., 0]$, and Tables 4.12 through 4.14 assume $\boldsymbol{\mu}_a = [\delta, \delta, \delta, ..., \delta]$. Again, for the same reasons in Section 4.3.2, Tables 4.9 through 4.14 are abbreviated from Tables B.7 through B.12 in Appendix B.

Table 4.9: τ = 50, p = 2, μ_a = (δ,0)

			MC1 CUSUM			MEWMA Actual Covariance Matrix		
Distance D_e	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15
0.00	0.00	200.38	192.53	195.66	197.57	208.01	202.18	199.56
		(1.92)	(1.88)	(2.05)	(2.02)	(2.33)	(2.11)	(2.04)
0.25	0.25	81.34	66.45	91.34	130.26	62.24	74.26	84.09
		(0.67)	(0.57)	(0.88)	(1.29)	(0.60)	(0.73)	(0.83)
0.50	0.50	31.36	25.98	31.19	55.63	23.60	26.36	30.36
		(0.22)	(0.17)	(0.26)	(0.54)	(0.18)	(0.21)	(0.26)
0.75	0.75	16.14	15.48	15.40	25.21	13.98	14.09	14.88
		(0.10)	(0.08)	(0.10)	(0.23)	(0.09)	(0.09)	(0.10)
1.00	1.00	10.34	11.02	9.78	12.85	9.92	9.38	9.52
		(0.06)	(0.05)	(0.06)	(0.10)	(0.06)	(0.05)	(0.06)
1.25	1.25	7.12	8.69	7.19	7.98	7.71	7.04	6.90
		(0.04)	(0.04)	(0.04)	(0.06)	(0.04)	(0.04)	(0.04)
1.50	1.50	5.31	7.13	5.71	5.57	6.29	5.68	5.40
		(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)
1.75	1.75	4.24	6.06	4.76	4.31	5.35	4.75	4.44
		(0.02)	(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)
2.00	2.00	3.42	5.31	4.12	3.49	4.66	4.12	3.80
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
2.25	2.25	2.86	4.75	3.64	2.96	4.19	3.62	3.35
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)
2.50	2.50	2.46	4.32	3.26	2.59	3.74	3.27	3.00
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)
2.75	2.75	2.14	3.94	2.98	2.32	3.46	2.96	2.72
3.00	3.00	1.89	3.64	2.75	2.10	3.17	2.76	2.50
3.25	3.25	1.70	3.40	2.55	1.91	2.94	2.53	2.32
3.50	3.50	1.54	3.17	2.39	1.78	2.78	2.40	2.16
3.75	3.75	1.42	2.98	2.26	1.66	2.83	2.30	2.07
4.00	4.00	1.31	2.83	2.16	1.55	2.47	2.12	1.93
4.25	4.25	1.23	2.68	2.05	1.46	2.36	2.03	1.83
4.50	4.50	1.17	2.56	1.96	1.40	2.25	1.92	1.76
4.75	4.75	1.11	2.45	1.87	1.34	2.14	1.86	1.68
5.00	5.00	1.06	2.36	1.79	1.29	2.08	1.77	1.61
		B = 6.66	h = 7.52	h = 4.78	h = 2.69	h4 = 7.86	h4 = 8.86	h4 = 9.39
RMI:		0.07	0.76	0.42	0.30	0.55	0.39	0.31

Table 4.9 illustrates when the MC1 and MEWMA are appropriately tuned they perform much better than the MMRC at detecting small shifts. However, when considering the entire range of shifts, the MMRC is superior as indicated by the RMI score. When compared with Table 4.2, the performance of both the MC1 and the

MEWMA decreases as D_e increases while the MMRC performance improves across the values of D_e.

Table 4.10: τ = 50, p = 3, μ_a = (δ,0,0)

			MC1 CUSUM			MEWMA Actual Covariance Matrix		
Distance D_e	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15
0.00	0.00	201.01	195.90	200.35	199.59	200.59	205.52	199.53
		(1.96)	(1.89)	(1.99)	(1.98)	(2.30)	(2.18)	(2.10)
0.25	0.25	90.41	73.56	100.07	141.04	65.65	85.34	98.44
		(0.74)	(0.62)	(0.96)	(1.40)	(0.65)	(0.85)	(0.97)
0.50	0.50	34.68	29.52	35.35	65.06	25.98	30.14	34.37
		(0.23)	(0.19)	(0.29)	(0.63)	(0.20)	(0.25)	(0.30)
0.75	0.75	18.44	17.90	17.80	28.73	14.98	15.72	16.90
		(0.11)	(0.10)	(0.12)	(0.26)	(0.10)	(0.11)	(0.12)
1.00	1.00	11.44	12.97	11.21	14.80	10.50	10.32	10.54
		(0.07)	(0.06)	(0.06)	(0.12)	(0.06)	(0.06)	(0.06)
1.25	1.25	7.96	10.14	8.24	9.08	8.14	7.71	7.53
		(0.04)	(0.05)	(0.04)	(0.06)	(0.05)	(0.04)	(0.04)
1.50	1.50	5.97	8.43	6.53	6.25	6.64	6.09	5.87
		(0.03)	(0.04)	(0.03)	(0.04)	(0.04)	(0.03)	(0.03)
1.75	1.75	4.64	7.21	5.52	4.79	5.68	5.12	4.81
		(0.02)	(0.03)	(0.02)	(0.03)	(0.03)	(0.02)	(0.02)
2.00	2.00	3.77	6.34	4.71	3.91	4.90	4.43	4.08
		(0.02)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
2.25	2.25	3.12	5.59	4.15	3.35	4.38	3.89	3.59
		(0.01)	(0.02)	(0.02)	(0.01)	(0.02)	(0.02)	(0.01)
2.50	2.50	2.69	5.06	3.78	2.93	3.98	3.47	3.15
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)
2.75	2.75	2.33	4.67	3.41	2.62	3.60	3.19	2.86
3.00	3.00	2.06	4.26	3.16	2.37	3.35	2.92	2.65
3.25	3.25	1.84	3.96	2.92	2.16	3.11	2.71	2.45
3.50	3.50	1.66	3.72	2.74	2.02	2.91	2.51	2.27
3.75	3.75	1.51	3.47	2.57	1.89	2.74	2.39	2.14
4.00	4.00	1.40	3.32	2.43	1.78	2.60	2.24	2.03
4.25	4.25	1.30	3.14	2.32	1.67	2.47	2.15	1.92
4.50	4.50	1.22	2.98	2.23	1.57	2.36	2.04	1.83
4.75	4.75	1.15	2.84	2.13	1.51	2.26	1.95	1.75
5.00	5.00	1.10	2.71	2.05	1.44	2.16	1.88	1.70
		B = 7.94	h = 8.79	h = 5.55	h = 3.15	h4 = 9.97	h4 = 11.11	h4 = 11.62
RMI:		0.08	0.92	0.51	0.37	0.53	0.39	0.31

Tables 4.10 through 4.14 at $\tau = 50$ are similar to tables 4.3 through 4.7 at $\tau = 0$. Note the difference between Tables 4.9 through 4.11 and Tables 4.12 through 4.14 is negligible when the only change is the out-of-control mean vector $\boldsymbol{\mu}_a$. As such, once can conclude the choice of $\boldsymbol{\mu}_a = [\delta, 0, 0, ..., 0]$ or $\boldsymbol{\mu}_a = [\delta, \delta, \delta, ..., \delta]$ has no effect on ARL performance. Furthermore, the increase of p shows the same tapering decrease in ARL performance seen in the previous section for all charts and tuning parameters.

There are two main differences between the $\tau = 50$ and $\tau = 0$ cases. The first is the consistently lower RMI of the MMRC to the other two charts. The other difference is as *p* increases, the properly tuned MC1 requires more observations to detect than the MMRC. For example, in Table 4.11, the MC1 ARL values for $k = \{0.25, 0.50, 1.00\}$ corresponding to $D_e = \{0.50, 1.00, 2.00\}$ are $\{48.30, 18.22, 6.24\}$ versus $\{48.09, 15.86, 5.12\}$ for the MMRC. However, Tables 4.9 through 4.14 continue to show a properly tuned MEWMA requires fewer observations to detect than the MMRC over a narrow range of D_e mean shift magnitudes. Overall, when the ARL performance is taken over the entire range of tested D_e mean shift magnitudes and $\tau = 50$, the MMRC is clearly the better chart.

Table 4.11: τ = 50, p = 10, μ_a = (δ,0,0,…,0)

			MC1 CUSUM			MEWMA Actual Covariance Matrix		
Distance D_e	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15
0.00	0.00	200.13	199.09	204.68	202.99	208.78	202.92	199.05
		(1.92)	(2.06)	(2.05)	(1.94)	(2.58)	(2.29)	(2.08)
0.25	0.25	113.42	105.73	127.87	159.73	88.63	112.94	130.05
		(0.92)	(0.91)	(1.22)	(1.57)	(1.02)	(1.23)	(1.37)
0.50	0.50	48.09	48.30	52.38	90.15	33.38	44.01	54.99
		(0.32)	(0.31)	(0.43)	(0.88)	(0.31)	(0.43)	(0.53)
0.75	0.75	25.27	29.72	27.01	43.13	18.69	21.23	25.74
		(0.15)	(0.16)	(0.18)	(0.39)	(0.15)	(0.17)	(0.21)
1.00	1.00	15.86	21.79	18.22	22.41	12.97	13.39	14.83
		(0.09)	(0.11)	(0.10)	(0.18)	(0.09)	(0.09)	(0.10)
1.25	1.25	11.11	17.25	13.79	13.40	9.89	9.54	10.18
		(0.06)	(0.08)	(0.07)	(0.09)	(0.06)	(0.06)	(0.06)
1.50	1.50	8.16	14.23	11.06	9.59	7.94	7.59	7.68
		(0.04)	(0.06)	(0.05)	(0.06)	(0.05)	(0.04)	(0.04)
1.75	1.75	6.33	12.12	9.37	7.48	6.80	6.20	6.16
		(0.03)	(0.05)	(0.04)	(0.04)	(0.04)	(0.03)	(0.03)
2.00	2.00	5.12	10.55	8.13	6.24	5.82	5.29	5.13
		(0.02)	(0.04)	(0.04)	(0.03)	(0.03)	(0.03)	(0.02)
2.25	2.25	4.28	9.41	7.12	5.32	5.13	4.61	4.43
		(0.02)	(0.04)	(0.03)	(0.02)	(0.03)	(0.02)	(0.02)
2.50	2.50	3.60	8.49	6.44	4.70	4.68	4.15	3.92
		(0.01)	(0.03)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)
2.75	2.75	3.10	7.68	5.82	4.20	4.25	3.71	3.51
3.00	3.00	2.71	7.06	5.35	3.79	3.92	3.43	3.21
3.25	3.25	2.43	6.56	4.96	3.48	3.62	3.10	2.93
3.50	3.50	2.18	6.10	4.60	3.21	3.42	2.94	2.74
3.75	3.75	1.98	5.75	4.31	3.03	3.19	2.74	2.55
4.00	4.00	1.80	5.38	4.04	2.83	3.01	2.59	2.40
4.25	4.25	1.65	5.12	3.84	2.67	2.87	2.46	2.27
4.50	4.50	1.53	4.85	3.60	2.53	2.73	2.33	2.15
4.75	4.75	1.42	4.60	3.44	2.43	2.61	2.23	2.06
5.00	5.00	1.33	4.40	3.30	2.31	2.50	2.15	1.97
		B = 14.75	h = 15.33	h = 9.58	h = 5.53	h4 = 21.97	h4 = 23.32	h4 = 23.89
RMI:		0.09	1.43	0.91	0.61	0.39	0.28	0.27

Table 4.12: τ = 50, p = 2, μ_a = (δ,δ)

			MC1 CUSUM			MEWMA Actual Covariance Matrix		
Distance D_e	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15
0.00	0.00	200.38	190.87	197.64	195.69	204.67	203.45	200.95
		(1.92)	(1.88)	(2.05)	(2.02)	(2.28)	(2.13)	(2.03)
0.25	0.18	83.17	65.93	90.19	128.61	62.02	74.49	83.54
		(1.94)	(0.55)	(0.86)	(1.27)	(0.59)	(0.73)	(0.83)
0.50	0.35	31.16	25.71	31.42	55.89	23.97	26.80	30.34
		(0.67)	(0.17)	(0.26)	(0.54)	(0.18)	(0.22)	(0.26)
0.75	0.53	16.39	15.50	15.42	25.15	14.09	14.03	15.11
		(0.21)	(0.08)	(0.10)	(0.23)	(0.09)	(0.09)	(0.11)
1.00	0.71	10.27	11.07	9.80	13.06	9.81	9.47	9.39
		(0.10)	(0.05)	(0.06)	(0.11)	(0.06)	(0.05)	(0.06)
1.25	0.88	7.20	8.61	7.22	8.06	7.78	7.06	6.86
		(0.06)	(0.04)	(0.04)	(0.06)	(0.04)	(0.04)	(0.04)
1.50	1.06	5.39	7.14	5.69	5.58	6.32	5.68	5.38
		(0.04)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)
1.75	1.24	4.20	6.11	4.78	4.29	5.40	4.78	4.45
		(0.03)	(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)
2.00	1.41	3.43	5.29	4.12	3.48	4.70	4.08	3.82
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
2.25	1.59	2.85	4.78	3.64	2.96	4.17	3.64	3.32
		(0.02)	(0.02)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)
2.50	1.77	2.44	4.31	3.26	2.60	3.77	3.28	2.98
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)
2.75	1.94	2.14	3.95	2.97	2.33	3.42	2.98	2.71
3.00	2.12	1.90	3.66	2.75	2.10	3.18	2.73	2.48
3.25	2.30	1.71	3.40	2.56	1.95	2.96	2.55	2.29
3.50	2.47	1.55	3.20	2.39	1.78	2.79	2.37	2.16
3.75	2.65	1.42	3.01	2.26	1.66	2.61	2.26	2.04
4.00	2.83	1.31	2.83	2.15	1.56	2.47	2.14	1.93
4.25	3.01	1.23	2.68	2.05	1.46	2.37	2.04	1.83
4.50	3.18	1.16	2.56	1.95	1.39	2.27	1.93	1.75
4.75	3.36	1.10	2.45	1.87	1.33	2.17	1.86	1.68
5.00	3.54	1.07	2.36	1.80	1.29	2.07	1.77	1.60
		B = 6.66	h = 7.52	h = 4.78	h = 2.69	h4 = 7.86	h4 = 8.86	h4 = 9.39
RMI:		0.07	0.76	0.42	0.30	0.56	0.39	0.30

Table 4.13: τ = 50, p = 3, μ_a = (δ,δ,δ)

			MC1 CUSUM			MEWMA Actual Covariance Matrix		
Distance D_e	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15
0.00	0.00	201.01	195.81	199.87	199.73	198.18	207.20	200.99
		(1.96)	(1.89)	(1.99)	(1.98)	(2.24)	(2.17)	(2.04)
0.25	0.14	89.82	73.56	100.07	141.04	66.64	84.19	96.97
		(0.74)	(0.62)	(0.96)	(1.40)	(0.67)	(0.83)	(0.94)
0.50	0.29	34.96	29.52	35.35	65.06	25.97	29.72	34.97
		(0.24)	(0.19)	(0.29)	(0.63)	(0.20)	(0.24)	(0.30)
0.75	0.43	18.14	17.90	17.80	28.73	14.98	16.01	16.94
		(0.11)	(0.10)	(0.12)	(0.26)	(0.10)	(0.11)	(0.12)
1.00	0.58	11.27	12.97	11.21	14.80	10.49	10.40	10.45
		(0.06)	(0.06)	(0.06)	(0.12)	(0.06)	(0.06)	(0.06)
1.25	0.72	7.94	10.14	8.24	9.08	8.13	7.71	7.47
		(0.04)	(0.05)	(0.04)	(0.06)	(0.05)	(0.04)	(0.04)
1.50	0.87	5.97	8.43	6.53	6.25	6.69	6.12	5.86
		(0.03)	(0.04)	(0.03)	(0.04)	(0.04)	(0.03)	(0.03)
1.75	1.01	4.63	7.21	5.52	4.79	5.65	5.11	4.80
		(0.02)	(0.03)	(0.02)	(0.03)	(0.03)	(0.02)	(0.02)
2.00	1.15	3.76	6.34	4.71	3.91	4.91	4.38	4.08
		(0.02)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
2.25	1.30	3.15	5.59	4.15	3.35	4.37	3.89	3.55
		(0.01)	(0.02)	(0.02)	(0.01)	(0.02)	(0.02)	(0.01)
2.50	1.44	2.68	5.06	3.78	2.93	3.95	3.49	3.19
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)
2.75	1.59	2.32	4.67	3.41	2.62	3.59	3.17	2.88
3.00	1.73	2.06	4.26	3.16	2.37	3.32	2.91	2.66
3.25	1.88	1.85	3.96	2.92	2.16	3.09	2.71	2.44
3.50	2.02	1.67	3.72	2.74	2.02	2.90	2.54	2.28
3.75	2.17	1.50	3.47	2.57	1.89	2.72	2.38	2.15
4.00	2.31	1.40	3.32	2.43	1.78	2.59	2.25	2.04
4.25	2.45	1.30	3.14	2.32	1.67	2.47	2.13	1.92
4.50	2.60	1.21	2.98	2.23	1.57	2.35	2.03	1.84
4.75	2.74	1.15	2.84	2.13	1.51	2.26	1.95	1.76
5.00	2.89	1.10	2.71	2.05	1.44	2.15	1.88	1.69
		B = 7.94	h = 8.79	h = 5.55	h = 3.15	h4 = 9.97	h4 = 11.11	h4 = 11.62
RMI:		0.07	0.91	0.51	0.37	0.52	0.38	0.31

Table 4.14: τ = 50, p = 10, μ_a = (δ,δ,δ,…,δ)

			MC1 CUSUM			MEWMA Actual Covariance Matrix		
Distance D_e	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15
0.00	0.00	200.13	202.12	205.45	202.83	198.32	202.91	203.95
		(1.92)	(2.06)	(2.05)	(1.94)	(2.47)	(2.30)	(2.13)
0.25	0.08	112.97	105.35	126.65	159.95	88.31	112.31	130.11
		(0.94)	(0.91)	(1.22)	(1.62)	(1.00)	(1.26)	(1.34)
0.50	0.16	47.67	48.31	52.22	89.59	33.94	43.65	54.49
		(0.31)	(0.32)	(0.43)	(0.88)	(0.31)	(0.42)	(0.52)
0.75	0.24	25.30	30.05	27.26	43.75	19.10	21.48	25.71
		(0.15)	(0.17)	(0.18)	(0.41)	(0.15)	(0.17)	(0.21)
1.00	0.32	15.80	21.86	18.21	22.61	12.78	13.25	14.74
		(0.09)	(0.11)	(0.10)	(0.18)	(0.09)	(0.09)	(0.10)
1.25	0.40	11.22	17.16	13.77	13.61	9.90	9.60	10.00
		(0.06)	(0.08)	(0.07)	(0.09)	(0.06)	(0.06)	(0.06)
1.50	0.47	8.18	14.28	11.15	9.64	7.98	7.44	7.50
		(0.04)	(0.06)	(0.05)	(0.06)	(0.05)	(0.04)	(0.04)
1.75	0.55	6.38	12.17	9.44	7.49	6.72	6.20	6.10
		(0.03)	(0.05)	(0.04)	(0.04)	(0.04)	(0.03)	(0.03)
2.00	0.63	5.13	10.62	8.06	6.23	5.87	5.30	5.15
		(0.02)	(0.04)	(0.04)	(0.03)	(0.03)	(0.03)	(0.02)
2.25	0.71	4.24	9.40	7.11	5.39	5.16	4.61	4.43
		(0.02)	(0.04)	(0.03)	(0.03)	(0.03)	(0.02)	(0.02)
2.50	0.79	3.60	8.48	6.43	4.68	4.60	4.12	3.90
		(0.02)	(0.03)	(0.03)	(0.02)	(0.03)	(0.02)	(0.02)
2.75	0.87	3.08	7.70	5.77	4.22	4.24	3.73	3.51
3.00	0.95	2.74	7.13	5.36	3.78	3.95	3.38	3.20
3.25	1.03	2.40	6.57	4.97	3.48	3.65	3.15	2.95
3.50	1.11	2.17	6.15	4.60	3.23	3.36	2.91	2.74
3.75	1.19	1.97	5.71	4.31	3.02	3.17	2.76	2.54
4.00	1.26	1.80	5.40	4.09	2.85	2.99	2.58	2.39
4.25	1.34	1.65	5.10	3.84	2.67	2.88	2.47	2.27
4.50	1.42	1.52	4.84	3.63	2.54	2.74	2.34	2.15
4.75	1.50	1.42	4.62	3.49	2.42	2.61	2.24	2.06
5.00	1.58	1.33	4.39	3.29	2.31	2.49	2.14	1.95
		B = 14.75	h = 15.33	h = 9.58	h = 5.53	h4 = 21.97	h4 = 23.32	h4 = 23.89
RMI:		0.09	1.43	0.91	0.61	0.39	0.28	0.26

4.5 MMRC Change Point Performance

Here the averages values of the MMRC's built-in change point estimator, $\hat{\tau}$, taken are displayed in Tables 4.15 and 4.16. These $\hat{\tau}$ averages were obtained during the ARL performance simulations in Sections 4.3 and 4.4 using the methodology in Section 4.2. The top two rows give the configuration for the number of variables, *p*, and the-out-of control mean, $\boldsymbol{\mu}_a$, used. The left most column of numbers corresponds to range of mean shifts D_e = {0.25,0.5,1.0,…,5}. The lower-right 20 by 6 matrix gives the simulated average values for $\hat{\tau}$ given a specific *p*, $\boldsymbol{\mu}_a$ and D_e.

Table 4.15: Change Point Estimator Performance when τ = 0

	p	2	3	10	2	3	10
	μ_a	(δ,0)	(δ,0,0)	(δ,0,…,0)	(δ,δ)	(δ,δ,δ)	(δ,δ,…,δ)
D_e	0.25	31	33	38	31	34	37
	0.5	19	21	30	19	22	29
	0.75	12	13	18	11	13	18
	1	7	8	12	8	8	12
	1.25	5	6	9	5	6	9
	1.5	4	4	6	4	4	6
	1.75	3	3	5	3	3	5
	2	2	3	4	2	2	4
	2.25	2	2	3	2	2	3
	2.5	1	2	2	1	2	2
	2.75	1	1	2	1	1	2
	3	1	1	2	1	1	2
	3.25	1	1	1	1	1	1
	3.5	1	1	1	1	1	1
	3.75	0	1	1	0	1	1
	4	0	0	1	0	0	1
	4.25	0	0	1	0	0	1
	4.5	0	0	1	0	0	1
	4.75	0	0	0	0	0	0
	5	0	0	0	0	0	0

In Table 4.15, when the process is out-of-control from the beginning, the MMRC has increased bias when detecting smaller shifts less than one. Also, the more variables there are the less accurate the change point becomes. Essentially, $\hat{\tau}$ is less biased when p is small and D_e is large.

Table 4.16: Change Point Estimator Performance when τ = 50

	p	2	3	10	2	3	10
	μ_a	(δ,0)	(δ,0,0)	(δ,0,…,0)	(δ,δ)	(δ,δ,δ)	(δ,δ,…,δ)
D_e	0.25	97	103	123	98	102	121
	0.5	59	60	65	59	60	66
	0.75	52	52	54	52	52	54
	1	50	51	51	50	50	51
	1.25	50	50	50	50	50	50
	1.5	50	50	50	50	50	50
	1.75	50	50	50	50	50	50
	2	50	50	50	50	50	50
	2.25	50	50	50	50	50	50
	2.5	50	50	50	50	50	50
	2.75	50	50	50	50	50	50
	3	50	50	50	50	50	50
	3.25	50	50	50	50	50	50
	3.5	50	50	50	50	50	50
	3.75	50	50	50	50	50	50
	4	50	50	50	50	50	50
	4.25	50	50	50	50	50	50
	4.5	50	50	50	50	50	50
	4.75	50	50	50	50	50	50
	5	50	50	50	50	50	50

For Table 4.16, when $\tau = 50$, the average value of $\hat{\tau}$ is exact for D_e from 1.25 to 5.00 and has a slight positive bias for 0.75 to 1.00. Although this bias is lessened when $\tau = 0$, the chart still has a substantial positive bias under small mean shift changes. Again, $\hat{\tau}$ is less biased when p is small and D_e is large.

4.6 MMRC CL (B) Regression Analysis

Like the UMRC, a regression analysis was used to provide B values resulting in estimated ARL_0 values from 50 to 300 and p from 2 to 10. Using the heuristic in Section 3.5, Figure 4.4 shows a 3D plot of the simulated B values. The figure shows input ARL_0 on the x-axis, input number of factors p on the y-axis, and simulated response B on the z-axis. Estimates for B were simulated at all ARL_0 = {50,100,150,200,250,300} versus p = {1,2,3,…,10} combinations for a total of 60 B estimates.

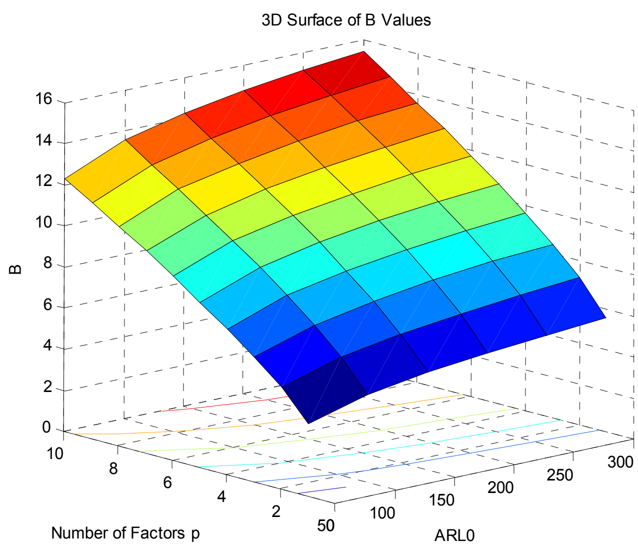

Figure 4.4: Simulated B Values

Clearly, an increase in either ARL_0 or p requires a larger B value. In fact, the plot resembles a rising ridge in RSM analysis. As a result, the following model is postulated:

$$\hat{B}(x_1, x_2) = c + x_1 + x_2 + x_1^2 + x_2^2 + x_1x_2 . \quad \textbf{(4.1)}$$

Here x_1 and x_2 are ARL$_0$ and *p* respectively. Using ordinary least squares, the following function was computed:

$$\hat{B}(x_1, x_2) = 1.30 + .0207x_1 + 1.32x_2 - .368E^{-4}x_1^2 - .338x_2^2 + .428E^{-3}x_1x_2 \quad \textbf{(4.2)}$$

When plotted, Equation **(4.2)** results in Figure 4.5 and the absolute difference between the simulation and regression approximation is documented in Table 4.17.

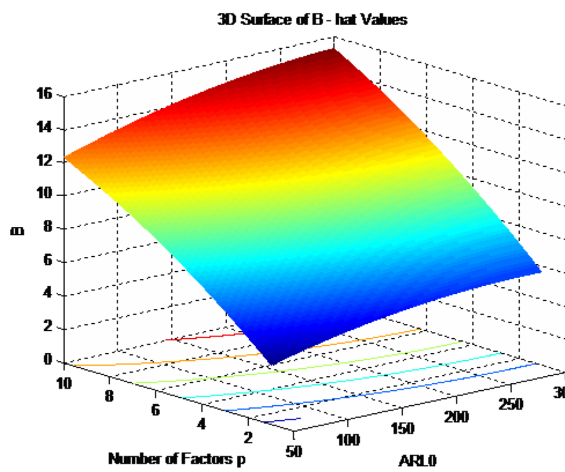

Figure 4.5: Regressed $\hat{B}$ Values

Table 4.17: Absolute Difference Between B and $\hat{B}$

ARL$_0$ \ *p*	1	2	3	4	5	6	7	8	9	10
50	0.075	0.107	0.060	0.003	0.128	0.148	0.194	0.171	0.118	0.044
100	0.050	0.175	0.190	0.171	0.117	0.048	0.046	0.084	0.134	0.270
150	0.183	0.067	0.107	0.057	0.018	0.026	0.022	0.005	0.051	0.157
200	0.271	0.013	0.050	0.013	0.054	0.101	0.127	0.104	0.042	0.042
250	0.240	0.032	0.065	0.037	0.044	0.103	0.151	0.131	0.099	0.029
300	0.081	0.188	0.191	0.160	0.076	0.004	0.028	0.048	0.009	0.069

While this approximation appears relatively accurate, the result is meaningless without checking the normality assumptions. The diagnostic plots in Figure 4.6 and Figure 4.7 revealed no serious violation(s) of the normality assumption.

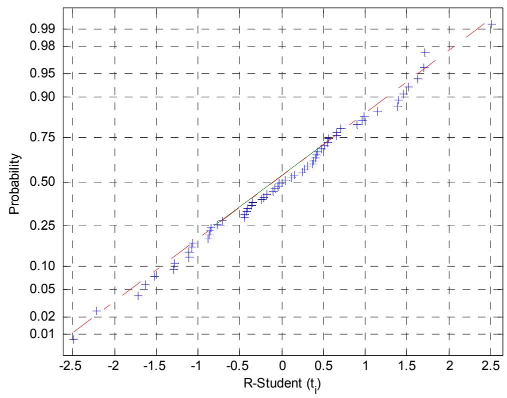

Figure 4.6: Normal Probability Plot

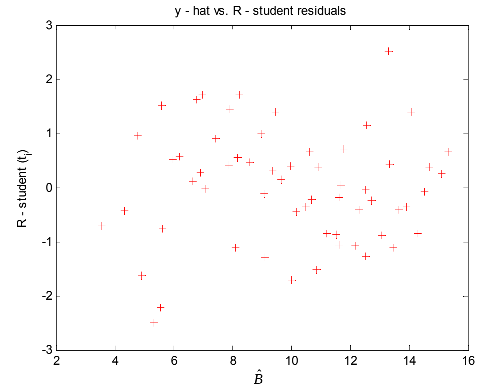

Figure 4.7: Regression vs. Residuals

4.7 MEWMA Warm-Up Effect

Unlike the MMRC and MC1, the MEWMA has a problem warming up. In other words, the chart takes time to build enough previous data, and while it is in this build-up phase, the ARL_0 changes if the CL, h_4, is held constant. This effect is more pronounced when the tuning parameter r is less than 0.10. For example, Figure 4.8 shows the change in ARL_0 with $r = 0.05$ and $p = 2$. Because of this warm-up effect, the ARL_0 increases with the actual covariance matrix and decreases with the steady-state covariance matrix. Furthermore, this is important because $r = 0.05$ provides superior ARL performance compared to the MMRC and MC1 when the change point, τ, is equal to zero (see Section 4.3). Finally, the number of variables $p \le 10$ had no impact on this warm-up effect.

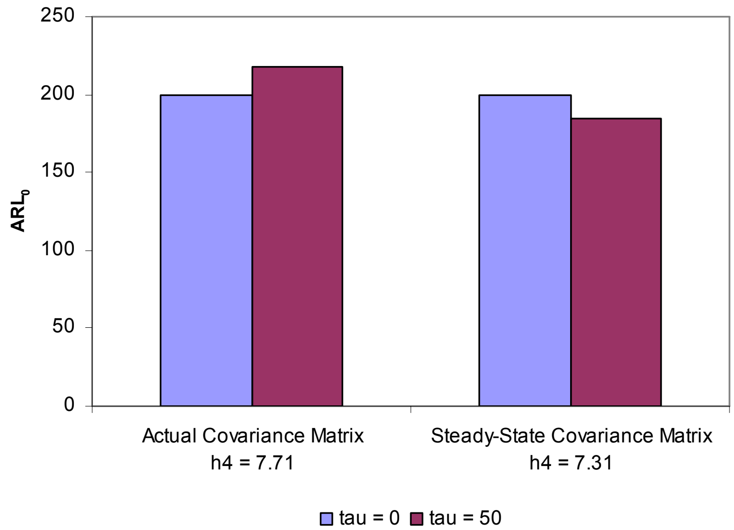

Figure 4.8: ARL_0 with *r* = .05 and *p* = 2

Figure 4.9 shows h_4 values corresponding to a specific τ obtained through simulation. The value of h_4 seems to rise until $\tau = 30$ where it averages 7.83 on the τ interval [30,400]. This means approximately 30 in-control runs are necessary for this particular chart to warm up.

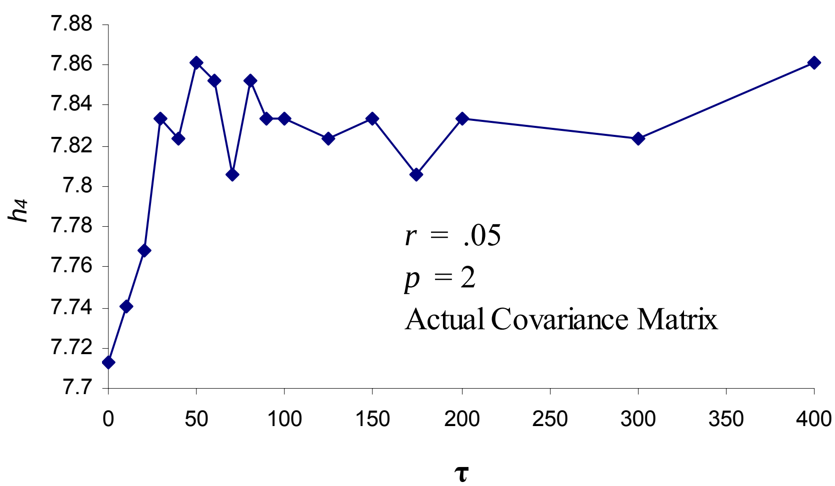

Figure 4.9: Warm-Up Effect of τ on h_4 for ARL_0 = 200

In their paper, Lowry *et al.* [8] do not refer to this problem and assume $\tau = 0$ for all of their h_4 calculations. Table 4.18 shows h_4 results tested at $\tau = 0$ and $\tau = 50$ for 15 different settings of *r* and *p* using the heuristic in Section 3.5. The table clearly indicates a lower *r* value results in a greater absolute difference between h_4 at $\tau = 0$ and $\tau = 50$. For comparison, the h_4 values by Lowry *et al.* are included which, in their paper they only include results for the *r* = 0.10 case.

Table 4.18: EWMA Inertia Comparison for ARL_0 = 200 with 10,000 Monte Carlo runs

		MEWMA Actual Covariance Matrix				MEWMA Steady-State Covariance Matrix			
r	*p*	τ = 0	τ = 50	Absolute Difference	Lowry et. al.	τ = 0	τ = 50	Absolute Difference	Lowry et. al.
0.05	2	7.71	7.86	0.15		7.31	7.60	0.30	
0.1	2	8.79	8.86	0.07	8.79	8.66	8.77	0.11	8.66
0.2	2	9.36	9.39	0.03		9.31	9.39	0.07	
0.5	2	10.50	10.50	0.00		10.46	10.42	0.05	
0.8	2	10.64	10.63	0.01		10.58	10.59	0.01	
0.05	3	9.82	9.97	0.15		9.39	9.66	0.27	
0.1	3	10.99	11.11	0.12	10.97	10.82	10.94	0.11	10.79
0.2	3	11.57	11.62	0.05		11.52	11.58	0.06	
0.5	3	12.71	12.70	0.01		12.69	12.71	0.02	
0.8	3	12.82	12.78	0.05		12.86	12.86	0.00	
0.05	10	21.41	21.97	0.56		20.76	21.06	0.31	
0.1	10	22.98	23.32	0.34	22.91	22.56	22.92	0.36	22.67
0.2	10	23.70	23.89	0.19		23.53	23.59	0.06	
0.5	10	25.07	25.00	0.07		25.06	25.04	0.02	
0.8	10	25.16	25.20	0.05		25.21	25.22	0.01	

For these reasons, Sections 4.3 and 4.4 used separate h_4 values corresponding to $\tau = 0$ and $\tau = 50$ respectively.

4.8 Conclusion

For the results obtained in this chapter, a Monte Carlo simulation for evaluating ARL performance was employed. By allowing for a flexible array of input variables, a QE can easily test simulation configurations not covered by this thesis. Furthermore, the code is adaptable enough to easily use a variety of different charts.

Using this simulation, the ARL performance evaluation yielded some interesting results. In the introduction, the following three questions were postulated with regard to ARL detection performance:

1. "What effect does the change point position have?"
2. "What effect does the number of out-of-control variables in the out-of-control vector have?"
3. "What effect does the number of variables in the process have?

For question one, the change point had a profound effect on the ARL performance. Overall, with two configuration exceptions, the MEWMA with $r = 0.05$ using the actual covariance matrix universally took the fewest observations to detect when the process was initially out-of-control. These two configuration exceptions are when the mean shift magnitude was 0.25 and 0.50, and the MEWMA took more observations to detect than the MC1 with $k = 0.25$. However, generally speaking, when $\tau = 50$, the MMRC took fewer observations to detect whereas the MC1 and MEWMA took more observations to detect. Furthermore, the MMRC's detection ability was not universally superior to the MC1 or MEWMA when $\tau = 50$. In fact the MEWMA, properly tuned to a specific mean shift range, provided superior ARL performance to the MMRC for both. Surprisingly, while the properly tuned MC1 at $\tau = 0$ bested the MMRC's detection ability for $D_e = \{0.50, 1.00, 2.00\}$, this was not the case when $\tau = 50$. Here, as p increased, the MC1 was shown to gradually lose its detection superiority to the MMRC. Furthermore, when taken across the entire range of tested mean shift magnitudes, the MMRC possesses the best ARL performance on the whole when $\tau = 50$. This is evidenced by the low RMI scores received by the MMRC.

For question two, whether one variable shifted or all variables shifted in the out-of-control mean vector made no significant difference in the results. This is likely due to both cases having the same distance, D_e, regardless of the number of shifted variables.

For question three, by and large, the more variables a process it has, the more observations a chart will take to detect a change. This is true for all three charts and all tested shift magnitudes. Fortunately this increase is not linear and appears to follow logarithmic scale.

Next, the ordinary least squares method was used to develop an estimator (Equation **(4.2)**) for the MMRC CL when given ARL_0 and the number of variables in the process. When this estimator was compared to the simulated values, the largest residual was 0.271 with an average residual of 0.094. As a result, the estimator provides accurate CL estimates for up to 10 variables and an ARL_0 on the interval [50,300].

Lastly, the MEWMA warm up effect was discussed. This means the MEWMA, given a static CL, was found to have different ARL_0 values depending on the actual change point. The results showed the effect was most pronounced when the tuning parameter, r, was less than 0.10. Since this effect would skew the ARL performance evaluations, different CLs for each of the change points were used in these evaluations.

V. Recommendations and Future Research

5.1 Introduction

This thesis presented the MMRC: a new multivariate mean shift control chart using multiple variables without the need for a tuning parameter. The word robust in the acronym MMRC refers to the chart's ability to perform well over a wide range of potential change magnitudes. Additionally, the MMRC is quite unique among multivariate control charts in its ability to provide the maximum likelihood estimate for the process change point immediately following a signal. In order to evaluate the current state of the art, this new chart's ARL performance was compared to two other existing multivariate charts, the MC1 and MEWMA.

In order to accomplish the goal of a multivariate magnitude robust control chart containing a change point estimator, a change point model using a log-likelihood-ratio test was employed. The change point model means the process is considered in-control until an unknown time τ where the mean vector has an immediate and sharp shift to an out-of-control state. In other words, all observations up to τ are in-control with the first out-of-control observation taken at $\tau + 1$. With the change point model in place, a log-likelihood-ratio test was used to detect a sudden shift in the mean vector. This log-likelihood-ratio test was maximized over all potential change points to obtain the greatest separation between the in-control and out-of-control mean vector. Once this test exceeds a predetermined CL, the process is considered out-of-control, and the change point maximizing the log-likelihood-ratio test becomes the MLE for τ, denoted as $\hat{\tau}$.

Additionally, $\hat{\tau}$ is used to create the MLE for the out-of-control mean vector. As a result of using log-likelihood-ratios, no tuning parameter is needed.

With no closed-form solution for the MMRC CL available, a simulation based heuristic was developed to generate CL values. This heuristic repeatedly increments and decrements candidate CLs until the resulting simulated ARL_0 values are within a specified tolerance limit. On a fairly modern PC, this heuristic takes only a few hours to output a control limit. To speed this process up further, this heuristic was run over a range of variables in the process and desired ARL_0 values. Taking this data and applying a regression model, a closed-form equation to obtain CL estimates was constructed.

Comparison was accomplished using Monte Carlo simulation to evaluate ARL performance. By setting the ARL_0 to 200 and the simulation run size to 10,000, direct comparison between the MMRC, MC1 and MEWMA was accomplished over two change points, three different number of variables and twenty mean shift distances. Additionally, the case where one variable in the out-of-control mean shifts was contrasted with the case where all variables in the out-of-control mean shift. Furthermore, the RMI was used to interpret the results. The RMI provides a singular measure of performance among several control charts over a range of tested change magnitudes. These results were presented in tabular form along with the RMI for each chart.

These simulation results showed the location of the change point, the number of variables and the magnitude of the mean shift were all significant in influencing ARL performance. A smaller mean shift magnitude and/or larger number of variables contributed to slower detection of the out-of-control process. The most profound difference came from the selection of the change point. In general, the performance of

the MMRC increased when the process was initially in-control for a period before going out-of-control and the MC1 and MEWMA was detected faster when the process was initially out-of-control. With regard to the RMI scores, the MC1 detected slowest over the entire range of tested sudden mean shifts, the MMRC detected quickest when $\tau = 50$ and the MEWMA detected quickest when $\tau = 0$. Only the variation of one variable versus all variables shifting in the out-of-control mean did not have an effect on ARL performance.

Finally, the average $\hat{\tau}$ results gathered from the ARL performance simulation were presented. These results showed $\hat{\tau}$ has less positive bias at $\tau = 50$ than when $\tau = 0$. Under both change points, $\hat{\tau}$ is highly biased for small out-of-control mean shifts and bias also increased as the number of variables increased.

5.2 Analysis Recommendation

While the MMRC possesses superior ARL performance when the process was in-control for a given time period, the chart clearly does not perform as well when the process is out-of-control initially. However, the MEWMA possesses superior ARL performance when the process is initially out-of-control. These results were shown in Sections 4.3 and 4.4. As a result, the practical recommendation use the MEWMA actual covariance matrix with $r = 0.05$ if the QE is uncertain whether the process will be in-control from the beginning. This situation could occur if one is basing the chart potentially faulty historical data. However, if the QE is highly confident his/her process will be initially in-control, use the MMRC for detection.

Also, when the MMRC detects a shift, the QE must keep in mind change point estimator $\hat{\tau}$ is positively biased when small shifts are detected. Conscious understanding of this limitation may prevent a costly search for causality.

5.3 Future Research Recommendations

This research is one of many possible approaches to the burgeoning yet incomplete field of multivariate control charts. This section will attempt to give suggestions and ideas for future research beyond the scope of this thesis.

To begin, if one relaxes one's assumptions, then one is almost always presented with interesting research problems. A major assumption of the MMRC is the assumption on normally distributed observations. While assuming normality is appropriate in many processes, it is not universally appropriate. For example, count data (*i.e.* product scratches) often follow a Poisson distribution. Applying a change point model and likelihood-ratio to other relevant distributions could generate new and improved control charts.

Another way to improve the MMRC is to consider autocorrelation. As the proposed methodology currently stands, it only considers cross-correlation between the variables. There is a need to extend the MMRC so it considers serial correlation.

In this thesis, all observations were pulled from a standard multivariate normal distribution. Performance when the normality assumption is violated was not considered. If charts were calculated using anything other than standard multivariate normal, then the results of this thesis are not applicable. Research into this area would illustrate the robustness of each chart to probability distribution violations.

Since the MMRC is a new chart, only one type of process change was considered and this is a sudden mean shift. However this is not the only type of process change. Linear or exponential trend shifts are often found in practice, however, these were not studied here. A study under these different assumptions would create a more complete picture of the MMRC's performance.

In addition to only one type of shift, only a static in-control mean of zero was used. Unfortunately, if the in-control state is periodic and/or sinusoidal, then one would likely have to deal with frequent false alarms. Thus, an investigation into different in-control states would extend the MMRC into these areas.

Appendix A

The following are the complete derivations from chapter three.

A.1 UMRC Derivation

$$f\left(x;\mu,\sigma\right)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right)$$

$$L_0\left(\boldsymbol{x}\right)=\prod_{t=1}^{T}\frac{1}{\sqrt{2\pi}\sigma_{\bar{x}}}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_0\right)^2\right)$$

$$L_a\left(\tau\mid\boldsymbol{x}\right)=\prod_{t=1}^{\tau}\frac{1}{\sqrt{2\pi}\sigma_{\bar{x}}}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_0\right)^2\right)\prod_{t=\tau+1}^{T}\frac{1}{\sqrt{2\pi}\sigma_{\bar{x}}}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_a\right)^2\right)$$

$$\begin{aligned}\frac{L_a\left(\tau\mid\overline{\mathrm{x}}\right)}{L_0\left(\overline{\mathrm{x}}\right)}&=\frac{\prod_{t=1}^{\tau}\frac{1}{\sqrt{2\pi}\sigma_{\bar{x}}}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_0\right)^2\right)*\prod_{t=\tau+1}^{T}\frac{1}{\sqrt{2\pi}\sigma_{\bar{x}}}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_a\right)^2\right)}{\prod_{t=1}^{T}\frac{1}{\sqrt{2\pi}\sigma_{\bar{x}}}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_0\right)^2\right)}\\
&=\frac{\prod_{t=\tau+1}^{T}\frac{1}{\sqrt{2\pi}\sigma_{\bar{x}}}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_a\right)^2\right)}{\prod_{t=\tau+1}^{T}\frac{1}{\sqrt{2\pi}\sigma_{\bar{x}}}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_0\right)^2\right)}\\
&=\frac{\prod_{t=\tau+1}^{T}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_a\right)^2\right)}{\prod_{t=\tau+1}^{T}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_0\right)^2\right)}\end{aligned}$$

$$\begin{aligned}R\left(\tau\mid\overline{\mathrm{x}}\right)&=\log_e\frac{L_a\left(\tau\mid\overline{\mathrm{x}}\right)}{L_0\left(\overline{\mathrm{x}}\right)}=\log_e\frac{\prod_{t=\tau+1}^{T}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_a\right)^2\right)}{\prod_{t=\tau+1}^{T}\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_0\right)^2\right)}\\
&=\sum_{t=\tau+1}^{T}\log_e\left(\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_a\right)^2\right)\right)-\sum_{t=\tau+1}^{T}\log_e\left(\exp\left(-\frac{1}{2\sigma_{\bar{x}}^2}\left(\bar{x}_t-\mu_0\right)^2\right)\right)\end{aligned}$$

$$= \frac{1}{2\sigma_{\bar{x}}^2}\left(\sum_{t=\tau+1}^{T} \left(\bar{x}_t - \mu_0\right)^2 - \sum_{t=\tau+1}^{T} \left(\bar{x}_t - \mu_a\right)^2 \right)$$

$$\hat{\mu}_a\left(\tau\right) = \bar{\bar{x}}_{T,\tau} = \frac{1}{T-\tau}\sum_{t=\tau+1}^{T} \bar{x}_i$$

$$\begin{aligned}
R_T = R\left(\tau \mid \bar{\boldsymbol{x}}\right) &= \max_{0 \le c < T} \frac{1}{2\sigma_{\bar{x}}^2}\left(\sum_{t=c+1}^{T} \left(\bar{x}_t - \mu_0\right)^2 - \sum_{t=c+1}^{T} \left(\bar{x}_t - \bar{\bar{x}}_{T,\tau}\right)^2 \right) \\
&= \max_{0 \le c < T} \frac{1}{2\sigma_{\bar{x}}^2} \sum_{t=c+1}^{T} \left(\left(\bar{x}_t - \mu_0\right)^2 - \left(\bar{x}_t - \bar{\bar{x}}_{T,c}\right)^2 \right) \\
&= \max_{0 \le c < T} \frac{1}{2\sigma_{\bar{x}}^2} \sum_{t=c+1}^{T} \left(\cancel{\bar{x}_t^2} - 2\mu_0\bar{x}_t + \mu_0^2 - \cancel{\bar{x}_t^2} + 2\bar{x}_t\bar{\bar{x}}_{T,c} - \bar{\bar{x}}_{T,c}^2 \right) \\
&= \max_{0 \le c < T} \frac{1}{2\sigma_{\bar{x}}^2} \left(-2\mu_0 \frac{\left(T-c\right)}{\left(T-c\right)} \sum_{t=c+1}^{T} \bar{x}_t + 2\bar{\bar{x}}_{T,c} \frac{\left(T-c\right)}{\left(T-c\right)} \sum_{t=c+1}^{T} \bar{x}_t + \sum_{t=c+1}^{T} \left(\mu_0^2 - \bar{\bar{x}}_{T,c}^2\right) \right) \\
&= \max_{0 \le c < T} \frac{1}{2\sigma_{\bar{x}}^2} \left(-2\mu_0 \left(T-c\right)\bar{\bar{x}}_{T,c} + \left(T-c\right)2\bar{\bar{x}}_{T,c}^2 + \left(T-c\right)\left(\mu_0^2 + 2\bar{x}_t\bar{\bar{x}}_{T,c} - \bar{\bar{x}}_{T,c}^2\right) \right) \\
&= \max_{0 \le c < T} \frac{T-c}{2\sigma_{\bar{x}}^2} \left(\bar{\bar{x}}_{T,c}^2 - 2\bar{\bar{x}}_{T,c}\mu_0 + \mu_0^2 \right) \\
&= \max_{0 \le c < T} \frac{T-c}{2\sigma_{\bar{x}}^2} \left(\bar{\bar{x}}_{T,c} - \mu_0 \right)^2
\end{aligned}$$

$$\hat{\tau} = \arg\max_{0 \le c < T} \frac{T-c}{2\sigma_{\bar{x}}^2} \left(\bar{\bar{x}}_{T,c} - \mu_0 \right)^2$$

$$\hat{\mu}_a\left(\hat{t}\right) = \bar{\bar{x}}_{T,\hat{\tau}} = \frac{T-\hat{\tau}}{2\sigma_{\bar{x}}^2} \sum_{i=\hat{\tau}+1}^{T} \bar{x}_i$$

A.2 MMRC Derivation

$$f\left(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{S}\right) = \frac{1}{\left(2\pi\right)^{p/2}\left|\boldsymbol{S}\right|^{1/2}} \exp\left\{ -\frac{1}{2}\left(\boldsymbol{x}-\boldsymbol{\mu}\right)' \boldsymbol{S}^{-1}\left(\boldsymbol{x}-\boldsymbol{\mu}\right) \right\}$$

$$L_0\left(\boldsymbol{X}\right) = \prod_{t=1}^{T} \frac{1}{\left(2\pi\right)^{p/2}\left|\boldsymbol{S}\right|^{1/2}} \exp\left\{ -\frac{1}{2}\left(\boldsymbol{x}_t-\boldsymbol{\mu_0}\right)' \boldsymbol{S}^{-1}\left(\boldsymbol{x}_t-\boldsymbol{\mu_0}\right) \right\}$$

$$L_a\left(\tau, \boldsymbol{\mu_a} \mid \boldsymbol{X}\right) = \prod_{t=1}^{\tau} \frac{1}{\left(2\pi\right)^{p/2} \left|\boldsymbol{S}\right|^{1/2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{x}_t - \boldsymbol{\mu_0}\right)' \boldsymbol{S}^{-1}\left(\boldsymbol{x}_t - \boldsymbol{\mu_0}\right)\right\}$$
$$\prod_{t=\tau+1}^{T} \frac{1}{\left(2\pi\right)^{p/2} \left|\boldsymbol{S}\right|^{1/2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{x}_t - \boldsymbol{\mu_a}\right)' \boldsymbol{S}^{-1}\left(\boldsymbol{x}_t - \boldsymbol{\mu_a}\right)\right\}$$

$$\frac{L_a\left(\tau, \boldsymbol{\mu_a} \mid \boldsymbol{X}\right)}{L_0\left(\boldsymbol{X}\right)} = \frac{\displaystyle\prod_{t=\tau+1}^{T} \frac{1}{\left(2\pi\right)^{p/2} \left|\boldsymbol{S}\right|^{1/2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{x}_t - \boldsymbol{\mu_a}\right)' \boldsymbol{S}^{-1}\left(\boldsymbol{x}_t - \boldsymbol{\mu_a}\right)\right\}}{\displaystyle\prod_{t=\tau+1}^{T} \frac{1}{\left(2\pi\right)^{p/2} \left|\boldsymbol{S}\right|^{1/2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{x}_t - \boldsymbol{\mu_0}\right)' \boldsymbol{S}^{-1}\left(\boldsymbol{x}_t - \boldsymbol{\mu_0}\right)\right\}}$$

$$R\left(\tau \mid \boldsymbol{X}\right) = \log_e \frac{L_a}{L_0} = \frac{1}{2}\sum_{t=\tau+1}^{T}\left[\left(\boldsymbol{x}_t - \boldsymbol{\mu_0}\right)' \boldsymbol{S}^{-1}\left(\boldsymbol{x}_t - \boldsymbol{\mu_0}\right) - \left(\boldsymbol{x}_t - \boldsymbol{\mu_0}\right)' \boldsymbol{S}^{-1}\left(\boldsymbol{x}_t - \boldsymbol{\mu_0}\right)\right]$$
$$= \frac{1}{2}\left[\left(T-\tau\right)\boldsymbol{\mu_0'}\boldsymbol{S}^{-1}\boldsymbol{\mu_0} - 2\boldsymbol{\mu_0'}\boldsymbol{S}^{-1}\sum_{t=\tau+1}^{T}\boldsymbol{x}_t + 2\boldsymbol{\mu_a'}\boldsymbol{S}^{-1}\sum_{t=\tau+1}^{T}\boldsymbol{x}_t - \left(T-\tau\right)\boldsymbol{\mu_a'}\boldsymbol{S}^{-1}\boldsymbol{\mu_a}\right]$$

$$\frac{\partial R}{\partial \boldsymbol{\mu_a}} = \boldsymbol{S}^{-1}\sum_{t=\tau+1}^{T}\boldsymbol{x}_t - \left(T-\tau\right)\boldsymbol{S}^{-1}\boldsymbol{\mu_a}$$

$$\boldsymbol{S}^{-1}\sum_{t=\tau+1}^{T}\boldsymbol{x}_t - \left(T-\tau\right)\boldsymbol{S}^{-1}\boldsymbol{\mu_a} = 0$$
$$\boldsymbol{S}^{-1}\frac{1}{T-\tau}\sum_{t=\tau+1}^{T}\boldsymbol{x}_t = \boldsymbol{S}^{-1}\boldsymbol{\mu_a}$$
$$\left(\boldsymbol{S}\right)\boldsymbol{S}^{-1}\frac{1}{T-\tau}\sum_{t=\tau+1}^{T}\boldsymbol{x}_t = \left(\boldsymbol{S}\right)\boldsymbol{S}^{-1}\boldsymbol{\mu_a}$$
$$\hat{\boldsymbol{\mu}}_{\boldsymbol{a}}\left(\tau\right) = \frac{1}{T-\tau}\sum_{t=\tau+1}^{T}\boldsymbol{x}_t = \overline{\boldsymbol{x}_{T,\tau}}$$

$$R_{\max} = R\left(\tau \mid \boldsymbol{X}\right) = \max_{0\le c<T}\frac{1}{2}\left[\left(T-c\right)\boldsymbol{\mu_0'}\boldsymbol{S}^{-1}\boldsymbol{\mu_0} - 2\boldsymbol{\mu_0'}\boldsymbol{S}^{-1}\sum_{t=c+1}^{T}\boldsymbol{x}_t\right.$$
$$\left. + 2\overline{\boldsymbol{x}'_{T,c}}\boldsymbol{S}^{-1}\sum_{t=c+1}^{T}\boldsymbol{x}_t - \left(T-c\right)\overline{\boldsymbol{x}'_{T,c}}\boldsymbol{S}^{-1}\overline{\boldsymbol{x}_{T,c}}\right]$$
$$= \max_{0\le c<T}\frac{T-c}{2}\left[\boldsymbol{\mu_0'}\boldsymbol{S}^{-1}\boldsymbol{\mu_0} - 2\boldsymbol{\mu_0'}\boldsymbol{S}^{-1}\overline{\boldsymbol{x}_{T,c}} + \cancel{2}\,\overline{\boldsymbol{x}'_{T,c}}\boldsymbol{S}^{-1}\overline{\boldsymbol{x}_{T,c}} - \cancel{\overline{\boldsymbol{x}'_{T,c}}\boldsymbol{S}^{-1}\overline{\boldsymbol{x}_{T,c}}}\right]$$
$$= \max_{0\le c<T}\frac{T-c}{2}\left[\boldsymbol{\mu_0'}\boldsymbol{S}^{-1}\boldsymbol{\mu_0} - 2\boldsymbol{\mu_0'}\boldsymbol{S}^{-1}\overline{\boldsymbol{x}_{T,c}} + \overline{\boldsymbol{x}'_{T,c}}\boldsymbol{S}^{-1}\overline{\boldsymbol{x}_{T,c}}\right]$$

$$= \max_{0 \le c < T} \frac{T-c}{2} \left[\left(\boldsymbol{\mu_0} - \overline{\boldsymbol{x}_{T,c}} \right)' \boldsymbol{S}^{-1} \left(\boldsymbol{\mu_0} - \overline{\boldsymbol{x}_{T,c}} \right) \right]$$

$$\hat{\tau} = \arg\max_{0 \le c < T} \frac{T-c}{2} \left[\left(\boldsymbol{\mu_0} - \overline{\boldsymbol{x}_{T,c}} \right)' \boldsymbol{S}^{-1} \left(\boldsymbol{\mu_0} - \overline{\boldsymbol{x}_{T,c}} \right) \right]$$

$$\hat{\boldsymbol{\mu}}_{\boldsymbol{a}} \left(\hat{\tau} \right) = \frac{1}{T - \hat{\tau}} \sum_{t=\hat{\tau}+1}^{T} \boldsymbol{x}_t = \overline{\boldsymbol{x}_{T,\hat{\tau}}}$$

Appendix B

The following pages contain the complete MMRC ARL data tables from chapter four.

Table B.1: $\tau = 0$, $p = 2$, $\mu_a = (\delta,0)$

			MC1 CUSUM			MEWMA Actual Covariance Matrix					MEWMA Steady-State Covariance Matrix				
Distance λ	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8
0.00	0.00	200.38	199.38	201.14	199.13	200.74	201.92	199.15	207.05	202.47	195.01	203.52	208.22	202.04	202.51
		(1.92)	(1.88)	(2.05)	(2.02)	(2.21)	(2.08)	(1.96)	(2.08)	(2.07)	(1.80)	(1.97)	(2.00)	(2.00)	(2.00)
0.25	0.25	87.20	66.86	93.06	128.14	58.93	73.78	85.22	138.03	165.17	65.51	77.21	88.83	140.81	160.86
		(0.69)	(0.54)	(0.87)	(1.28)	(0.56)	(0.71)	(0.82)	(1.39)	(1.66)	(0.51)	(0.68)	(0.83)	(1.37)	(1.60)
0.50	0.50	34.15	25.38	31.23	56.27	20.99	25.17	29.10	63.83	96.68	26.58	28.47	31.30	63.63	95.46
		(0.22)	(0.15)	(0.26)	(0.55)	(0.17)	(0.20)	(0.25)	(0.63)	(0.96)	(0.15)	(0.20)	(0.25)	(0.61)	(0.93)
0.75	0.75	17.89	14.79	15.08	24.96	10.78	12.66	13.96	29.43	52.19	15.76	15.26	15.72	30.04	51.69
		(0.11)	(0.07)	(0.10)	(0.23)	(0.08)	(0.09)	(0.11)	(0.28)	(0.50)	(0.07)	(0.08)	(0.10)	(0.28)	(0.50)
1.00	1.00	11.09	10.31	9.26	12.90	6.82	7.79	8.53	15.80	28.52	11.17	10.10	10.10	15.83	28.66
		(0.06)	(0.04)	(0.05)	(0.11)	(0.05)	(0.05)	(0.06)	(0.14)	(0.28)	(0.04)	(0.05)	(0.05)	(0.14)	(0.27)
1.25	1.25	7.63	7.98	6.74	7.79	4.77	5.41	5.71	9.25	16.48	8.67	7.65	7.30	9.46	16.71
		(0.04)	(0.03)	(0.03)	(0.06)	(0.03)	(0.03)	(0.04)	(0.08)	(0.15)	(0.03)	(0.03)	(0.03)	(0.07)	(0.16)
1.50	1.50	5.67	6.51	5.20	5.31	3.63	4.04	4.30	6.04	10.30	7.12	6.10	5.66	6.40	10.32
		(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)	(0.03)	(0.04)	(0.09)	(0.02)	(0.02)	(0.02)	(0.05)	(0.09)
1.75	1.75	4.44	5.49	4.34	3.97	2.90	3.21	3.35	4.40	6.75	6.03	5.10	4.70	4.62	6.70
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.03)	(0.06)	(0.02)	(0.02)	(0.02)	(0.03)	(0.06)
2.00	2.00	3.62	4.78	3.69	3.21	2.36	2.62	2.73	3.35	4.77	5.25	4.41	4.02	3.62	4.76
		(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.04)	(0.01)	(0.01)	(0.01)	(0.02)	(0.04)
2.25	2.25	3.00	4.27	3.25	2.72	2.00	2.19	2.28	2.72	3.52	4.68	3.89	3.53	2.97	3.56
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.03)	(0.01)	(0.01)	(0.01)	(0.01)	(0.03)
2.50	2.50	2.55	3.84	2.90	2.32	1.75	1.88	1.98	2.29	2.71	4.22	3.50	3.13	2.52	2.79
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
2.75	2.75	2.21	3.52	2.64	2.08	1.55	1.67	1.73	1.95	2.26	3.84	3.18	2.84	2.19	2.26
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.00	3.00	1.96	3.25	2.42	1.87	1.40	1.49	1.55	1.70	1.89	3.54	2.93	2.61	1.94	1.91
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.25	3.25	1.74	3.02	2.26	1.69	1.29	1.37	1.40	1.53	1.63	3.30	2.71	2.43	1.75	1.64
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.50	3.50	1.58	2.82	2.12	1.56	1.20	1.26	1.30	1.38	1.44	3.09	2.51	2.25	1.60	1.46
		(0.01)	(0.01)		(0.01)			(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.00)	(0.01)	(0.01)
3.75	3.75	1.45	2.64	2.01	1.44	1.13	1.18	1.21	1.28	1.31	2.91	2.36	2.14	1.46	1.33
		(0.01)	(0.01)		(0.01)					(0.01)	(0.01)	(0.01)	(0.00)	(0.01)	(0.01)
4.00	4.00	1.32	2.48	1.93	1.34	1.09	1.12	1.14	1.19	1.21	2.74	2.24	2.05	1.36	1.23
4.25	4.25	1.24	2.34	1.83	1.25	1.05	1.08	1.09	1.13	1.14	2.59	2.12	1.97	1.26	1.16
4.50	4.50	1.17	2.22	1.76	1.18	1.03	1.05	1.06	1.09	1.09	2.44	2.07	1.91	1.18	1.10
4.75	4.75	1.11	2.13	1.68	1.12	1.02	1.02	1.03	1.05	1.05	2.31	2.01	1.83	1.13	1.06
5.00	5.00	1.07	2.06	1.58	1.08	1.01	1.02	1.02	1.03	1.03	2.19	1.97	1.76	1.08	1.04
		B = 6.66	h = 7.52	h = 4.78	h = 2.69	h4 = 7.71	h4 = 8.79	h4 = 9.36	h4 = 10.5	h4 = 10.64	h4 = 7.31	h4 = 8.66	h4 = 9.31	h4 = 10.46	h4 = 10.58
RMI:		0.40	0.97	0.61	0.49	0.00	0.09	0.15	0.54	1.10	1.14	0.84	0.72	0.64	1.11

Table B.2: τ = 0, p = 3, μ_a = (δ,0,0)

			MC1 CUSUM			MEWMA Actual Covariance Matrix					MEWMA Steady-State Covariance Matrix				
Distance λ	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8
0.00	0.00	201.01	200.34	201.59	199.93	202.34	202.68	199.02	199.73	196.48	202.25	203.64	205.36	198.15	198.94
		(1.96)	(1.89)	(1.99)	(1.98)	(2.17)	(2.04)	(2.00)	(2.01)	(1.96)	(1.89)	(1.94)	(1.99)	(1.98)	(1.95)
0.25	0.25	95.05	71.77	99.77	139.77	67.33	84.09	94.74	149.18	169.34	74.33	88.58	100.64	151.61	172.57
		(0.74)	(0.59)	(0.96)	(1.37)	(0.63)	(0.81)	(0.92)	(1.50)	(1.69)	(0.57)	(0.78)	(0.94)	(1.47)	(1.73)
0.50	0.50	37.43	27.86	34.41	63.91	23.53	28.77	34.02	75.96	108.73	29.77	31.89	35.99	76.03	111.94
		(0.24)	(0.17)	(0.29)	(0.61)	(0.19)	(0.24)	(0.29)	(0.74)	(1.08)	(0.17)	(0.23)	(0.29)	(0.73)	(1.10)
0.75	0.75	19.72	16.16	16.46	28.46	12.38	14.43	16.00	36.55	63.34	17.61	17.00	17.95	37.07	64.64
		(0.12)	(0.07)	(0.11)	(0.26)	(0.09)	(0.10)	(0.12)	(0.35)	(0.62)	(0.08)	(0.09)	(0.12)	(0.35)	(0.64)
1.00	1.00	12.24	11.39	10.16	14.29	7.67	8.75	9.57	19.11	35.61	12.46	11.29	11.26	19.26	35.82
		(0.07)	(0.04)	(0.05)	(0.12)	(0.05)	(0.06)	(0.06)	(0.17)	(0.35)	(0.04)	(0.05)	(0.06)	(0.17)	(0.35)
1.25	1.25	8.46	8.82	7.30	8.47	5.38	6.04	6.48	10.83	21.10	9.69	8.36	8.10	11.24	21.30
		(0.04)	(0.03)	(0.03)	(0.06)	(0.03)	(0.04)	(0.04)	(0.09)	(0.20)	(0.03)	(0.03)	(0.04)	(0.09)	(0.20)
1.50	1.50	6.27	7.27	5.77	5.80	3.99	4.46	4.76	7.12	12.52	7.97	6.76	6.27	7.38	12.89
		(0.03)	(0.02)	(0.02)	(0.04)	(0.02)	(0.03)	(0.03)	(0.06)	(0.12)	(0.02)	(0.02)	(0.02)	(0.05)	(0.12)
1.75	1.75	4.88	6.15	4.73	4.30	3.18	3.56	3.70	4.96	8.29	6.73	5.61	5.15	5.30	8.40
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.03)	(0.07)	(0.02)	(0.02)	(0.02)	(0.03)	(0.07)
2.00	2.00	3.93	5.39	4.05	3.48	2.59	2.86	3.02	3.85	5.68	5.82	4.83	4.41	4.07	5.73
		(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.02)	(0.05)	(0.01)	(0.01)	(0.01)	(0.02)	(0.05)
2.25	2.25	3.29	4.81	3.57	2.91	2.19	2.41	2.52	3.04	4.18	5.20	4.29	3.83	3.30	4.24
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.03)	(0.01)	(0.01)	(0.01)	(0.02)	(0.03)
2.50	2.50	2.78	4.32	3.20	2.54	1.93	2.07	2.16	2.49	3.22	4.68	3.83	3.42	2.76	3.24
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
2.75	2.75	2.42	3.94	2.89	2.24	1.67	1.82	1.89	2.12	2.54	4.27	3.47	3.09	2.39	2.59
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
3.00	3.00	2.12	3.64	2.65	2.03	1.50	1.62	1.68	1.85	2.10	3.94	3.21	2.84	2.13	2.15
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.25	3.25	1.88	3.38	2.46	1.86	1.37	1.47	1.52	1.64	1.78	3.64	2.96	2.62	1.92	1.85
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.50	3.50	1.71	3.17	2.30	1.71	1.27	1.34	1.38	1.49	1.58	3.40	2.77	2.43	1.74	1.62
		(0.01)	(0.01)	(0.01)	(0.01)		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.75	3.75	1.55	2.99	2.18	1.59	1.19	1.24	1.27	1.35	1.40	3.21	2.58	2.28	1.60	1.44
		(0.01)	(0.01)		(0.01)				(0.01)	(0.01)	(0.01)	(0.01)	(0.00)	(0.01)	(0.01)
4.00	4.00	1.43	2.83	2.08	1.48	1.13	1.17	1.20	1.26	1.28	3.03	2.44	2.17	1.48	1.31
4.25	4.25	1.32	2.68	2.00	1.37	1.08	1.12	1.14	1.18	1.19	2.89	2.29	2.08	1.36	1.21
4.50	4.50	1.23	2.53	1.94	1.29	1.05	1.07	1.09	1.12	1.13	2.75	2.17	2.02	1.27	1.14
4.75	4.75	1.16	2.38	1.88	1.21	1.03	1.04	1.06	1.08	1.08	2.62	2.10	1.97	1.20	1.09
5.00	5.00	1.11	2.26	1.82	1.14	1.02	1.03	1.03	1.05	1.05	2.46	2.05	1.91	1.14	1.06
		B = 7.94	h = 8.79	h = 5.55	h = 3.15	h4 = 9.82	h4 = 10.99	h4 = 11.57	h4 = 12.71	h4 = 12.82	h4 = 9.39	h4 = 10.82	h4 = 11.52	h4 = 12.69	h4 = 12.86
RMI:		0.40	1.06	0.64	0.51	0.00	0.09	0.15	0.59	1.22	1.23	0.87	0.75	0.71	1.26

Table B.3: $\tau = 0$, $p = 10$, $\mu_a = (\delta,0,0,\ldots,0)$

			MC1 CUSUM			MEWMA Actual Covariance Matrix					MEWMA Steady-State Covariance Matrix				
Distance λ	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8
0.00	0.00	200.13	200.58	201.14	198.96	205.21	204.17	198.00	199.75	198.71	201.22	193.79	198.77	202.96	200.72
		(1.92)	(2.06)	(2.05)	(1.94)	(2.16)	(2.08)	(2.02)	(1.96)	(2.00)	(1.86)	(1.84)	(1.91)	(2.03)	(1.99)
0.25	0.25	118.79	84.20	122.42	158.52	93.34	118.13	128.21	174.76	180.41	100.90	114.29	131.65	175.30	184.88
		(0.94)	(0.78)	(1.19)	(1.62)	(0.92)	(1.18)	(1.28)	(1.74)	(1.78)	(0.78)	(1.00)	(1.25)	(1.70)	(1.88)
0.50	0.50	51.71	33.47	44.65	88.47	34.85	44.94	54.58	112.47	145.38	42.65	48.32	56.82	115.58	147.06
		(0.33)	(0.17)	(0.41)	(0.89)	(0.28)	(0.39)	(0.50)	(1.14)	(1.46)	(0.25)	(0.36)	(0.48)	(1.15)	(1.44)
0.75	0.75	27.66	21.56	19.69	40.30	17.78	21.62	25.60	65.92	104.05	24.97	24.79	27.38	67.90	104.40
		(0.15)	(0.08)	(0.13)	(0.39)	(0.13)	(0.16)	(0.21)	(0.65)	(1.03)	(0.10)	(0.15)	(0.19)	(0.65)	(1.02)
1.00	1.00	17.16	16.09	12.64	19.32	11.02	12.74	14.38	36.96	68.84	17.43	15.67	16.35	37.60	70.11
		(0.09)	(0.05)	(0.06)	(0.17)	(0.07)	(0.08)	(0.10)	(0.35)	(0.68)	(0.06)	(0.07)	(0.09)	(0.35)	(0.69)
1.25	1.25	11.85	12.86	9.50	10.67	7.73	8.69	9.41	20.80	43.60	13.49	11.62	11.31	21.23	44.95
		(0.06)	(0.03)	(0.03)	(0.08)	(0.05)	(0.05)	(0.06)	(0.19)	(0.42)	(0.04)	(0.04)	(0.05)	(0.19)	(0.44)
1.50	1.50	8.68	10.74	7.74	7.00	5.70	6.39	6.84	13.01	27.65	11.06	9.21	8.61	13.33	28.05
		(0.04)	(0.02)	(0.02)	(0.04)	(0.03)	(0.04)	(0.04)	(0.11)	(0.27)	(0.03)	(0.03)	(0.03)	(0.11)	(0.26)
1.75	1.75	6.74	9.32	6.53	5.30	4.43	4.96	5.20	8.58	17.76	9.40	7.60	6.97	9.02	18.29
		(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.03)	(0.03)	(0.07)	(0.17)	(0.02)	(0.02)	(0.02)	(0.07)	(0.17)
2.00	2.00	5.40	8.19	5.69	4.37	3.60	3.94	4.17	6.08	11.81	8.15	6.53	5.93	6.40	12.10
		(0.02)	(0.02)	(0.01)	(0.02)	(0.02)	(0.02)	(0.02)	(0.04)	(0.11)	(0.02)	(0.02)	(0.02)	(0.04)	(0.11)
2.25	2.25	4.46	7.31	5.07	3.69	3.00	3.28	3.44	4.59	8.18	7.22	5.75	5.11	4.97	8.28
		(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)	(0.03)	(0.07)	(0.01)	(0.01)	(0.01)	(0.03)	(0.07)
2.50	2.50	3.77	6.66	4.54	3.29	2.58	2.81	2.90	3.65	5.84	6.50	5.15	4.55	4.03	6.05
		(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.05)	(0.01)	(0.01)	(0.01)	(0.02)	(0.05)
2.75	2.75	3.23	6.10	4.17	2.95	2.23	2.41	2.54	2.99	4.39	5.93	4.64	4.10	3.34	4.49
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.03)	(0.01)	(0.01)	(0.01)	(0.02)	(0.03)
3.00	3.00	2.83	5.61	3.83	2.69	1.99	2.14	2.22	2.56	3.39	5.44	4.28	3.74	2.89	3.55
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.03)
3.25	3.25	2.50	5.22	3.57	2.47	1.78	1.89	1.98	2.21	2.77	5.05	3.95	3.45	2.56	2.86
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
3.50	3.50	2.22	4.87	3.35	2.30	1.61	1.71	1.79	1.95	2.32	4.71	3.66	3.21	2.31	2.39
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.75	3.75	2.01	4.58	3.15	2.18	1.47	1.57	1.62	1.75	1.97	4.42	3.43	3.00	2.12	2.04
		(0.01)	(0.01)	(0.01)		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
4.00	4.00	1.82	4.33	2.99	2.08	1.35	1.45	1.48	1.60	1.72	4.17	3.25	2.82	1.96	1.79
4.25	4.25	1.67	4.11	2.84	2.00	1.27	1.34	1.38	1.45	1.54	3.94	3.08	2.67	1.82	1.60
4.50	4.50	1.55	3.92	2.70	1.92	1.21	1.26	1.29	1.35	1.39	3.74	2.95	2.52	1.69	1.43
4.75	4.75	1.44	3.74	2.56	1.85	1.14	1.19	1.21	1.26	1.28	3.55	2.82	2.38	1.59	1.33
5.00	5.00	1.34	3.56	2.42	1.78	1.09	1.14	1.15	1.18	1.20	3.37	2.68	2.25	1.49	1.24
		B = 14.75	h = 15.33	h = 9.58	h = 5.53	h4 = 21.41	h4 = 22.98	h4 = 23.7	h4 = 25.07	h4 = 25.16	h4 = 20.76	h4 = 22.56	h4 = 23.53	h4 = 25.06	h4 = 25.21
RMI:		0.43	1.42	0.77	0.55	0.01	0.12	0.19	0.80	1.76	1.41	0.97	0.80	0.96	1.83

Table B.4: $\tau = 0$, $p = 2$, $\mu_a = (\delta,\delta)$

			MC1 CUSUM			MEWMA Actual Covariance Matrix					MEWMA Steady-State Covariance Matrix				
Distance λ	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8
0.00	0.00	200.38	199.38	201.14	199.13	202.45	196.10	200.42	203.64	203.87	197.82	202.95	201.97	199.65	199.12
		(1.92)	(1.88)	(2.05)	(2.02)	(2.17)	(2.01)	(2.02)	(2.06)	(2.03)	(1.83)	(1.94)	(1.97)	(1.98)	(1.95)
0.25	0.18	87.99	67.42	92.54	129.03	58.86	73.48	84.35	136.93	165.32	65.74	77.95	89.51	138.41	157.97
		(0.68)	(0.55)	(0.87)	(1.28)	(0.56)	(0.70)	(0.83)	(1.37)	(1.63)	(0.51)	(0.69)	(0.82)	(1.36)	(1.56)
0.50	0.35	33.85	25.44	31.00	56.59	20.44	25.00	28.91	65.26	96.69	26.20	28.04	30.98	63.87	94.80
		(0.22)	(0.15)	(0.25)	(0.55)	(0.16)	(0.21)	(0.25)	(0.64)	(0.95)	(0.15)	(0.20)	(0.24)	(0.63)	(0.94)
0.75	0.53	17.65	14.74	15.11	25.11	10.90	12.78	13.85	30.21	52.73	15.66	15.16	15.93	30.13	51.25
		(0.11)	(0.07)	(0.10)	(0.23)	(0.08)	(0.09)	(0.10)	(0.28)	(0.53)	(0.07)	(0.08)	(0.10)	(0.28)	(0.50)
1.00	0.71	11.18	10.31	9.23	12.89	6.84	7.78	8.46	15.84	29.11	11.11	10.15	10.04	16.00	28.36
		(0.06)	(0.04)	(0.05)	(0.11)	(0.05)	(0.05)	(0.06)	(0.14)	(0.28)	(0.04)	(0.05)	(0.05)	(0.14)	(0.28)
1.25	0.88	7.64	7.97	6.76	7.75	4.83	5.41	5.77	9.15	16.65	8.65	7.59	7.22	9.56	16.50
		(0.04)	(0.03)	(0.03)	(0.06)	(0.03)	(0.03)	(0.04)	(0.08)	(0.16)	(0.03)	(0.03)	(0.03)	(0.07)	(0.15)
1.50	1.06	5.68	6.51	5.23	5.32	3.63	4.00	4.28	6.08	10.29	7.10	6.13	5.71	6.44	10.15
		(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)	(0.02)	(0.04)	(0.09)	(0.02)	(0.02)	(0.02)	(0.05)	(0.09)
1.75	1.24	4.41	5.50	4.35	3.98	2.87	3.18	3.36	4.34	6.73	6.04	5.11	4.74	4.64	6.78
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.03)	(0.06)	(0.02)	(0.02)	(0.02)	(0.03)	(0.06)
2.00	1.41	3.59	4.80	3.73	3.20	2.34	2.62	2.74	3.36	4.81	5.25	4.41	4.01	3.60	4.73
		(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.04)	(0.01)	(0.01)	(0.01)	(0.02)	(0.04)
2.25	1.59	2.99	4.27	3.25	2.72	1.98	2.21	2.28	2.73	3.49	4.66	3.89	3.51	2.95	3.58
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.03)	(0.01)	(0.01)	(0.01)	(0.01)	(0.03)
2.50	1.77	2.56	3.84	2.90	2.33	1.74	1.89	1.97	2.26	2.75	4.21	3.50	3.15	2.50	2.77
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
2.75	1.94	2.21	3.52	2.64	2.08	1.54	1.67	1.72	1.94	2.22	3.85	3.18	2.84	2.17	2.23
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.00	2.12	1.95	3.25	2.43	1.88	1.40	1.48	1.55	1.70	1.88	3.54	2.92	2.61	1.95	1.90
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.25	2.30	1.75	3.02	2.26	1.70	1.27	1.36	1.41	1.53	1.62	3.29	2.72	2.43	1.76	1.66
		(0.01)	(0.01)	(0.01)	(0.01)		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.50	2.47	1.57	2.82	2.13	1.56	1.20	1.27	1.30	1.38	1.45	3.08	2.52	2.26	1.60	1.46
		(0.01)	(0.01)		(0.01)			(0.01)	(0.01)	(0.01)	(0.01)	(0.01)		(0.01)	(0.01)
3.75	2.65	1.45	2.64	2.02	1.44	1.13	1.18	1.21	1.27	1.32	2.91	2.37	2.14	1.47	1.32
		(0.01)	(0.01)		(0.01)					(0.01)	(0.01)	(0.01)		(0.01)	(0.01)
4.00	2.83	1.33	2.49	1.92	1.34	1.09	1.12	1.14	1.19	1.21	2.75	2.23	2.05	1.36	1.22
4.25	3.01	1.24	2.33	1.84	1.25	1.05	1.08	1.09	1.13	1.14	2.59	2.14	1.97	1.26	1.15
4.50	3.18	1.17	2.22	1.76	1.18	1.03	1.05	1.06	1.09	1.09	2.43	2.06	1.91	1.18	1.10
4.75	3.36	1.11	2.13	1.67	1.12	1.02	1.03	1.03	1.05	1.06	2.30	2.01	1.83	1.12	1.06
5.00	3.54	1.08	2.07	1.57	1.08	1.01	1.02	1.02	1.03	1.03	2.18	1.97	1.77	1.08	1.04
		B = 6.66	h = 7.52	h = 4.78	h = 2.69	h4 = 7.71	h4 = 8.79	h4 = 9.36	h4 = 10.5	h4 = 10.64	h4 = 7.31	h4 = 8.66	h4 = 9.31	h4 = 10.46	h4 = 10.58
RMI:		0.41	0.98	0.61	0.50	0.00	0.09	0.15	0.55	1.12	1.14	0.85	0.73	0.65	1.10

Table B.5: $\tau = 0$, $p = 3$, $\mu_a = (\delta,\delta,\delta)$

			MC1 CUSUM			MEWMA Actual Covariance Matrix					MEWMA Steady-State Covariance Matrix				
Distance λ	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8
0.00	0.00	201.01	200.34	201.59	199.93	203.24	201.85	203.96	201.63	202.30	201.76	205.44	206.70	200.42	202.79
		(1.96)	(1.89)	(1.99)	(1.98)	(2.18)	(2.09)	(2.10)	(2.06)	(2.05)	(1.87)	(1.99)	(2.01)	(2.00)	(2.01)
0.25	0.14	96.21	73.63	99.94	141.63	66.91	82.28	97.73	148.89	168.41	73.17	87.62	100.10	148.73	170.66
		(0.75)	(0.61)	(0.95)	(1.42)	(0.62)	(0.78)	(0.97)	(1.48)	(1.66)	(0.56)	(0.77)	(0.93)	(1.46)	(1.70)
0.50	0.29	37.94	27.88	34.60	63.66	23.47	28.97	34.30	74.82	109.51	29.95	32.27	36.52	74.48	109.70
		(0.25)	(0.17)	(0.29)	(0.61)	(0.19)	(0.24)	(0.30)	(0.73)	(1.08)	(0.17)	(0.23)	(0.29)	(0.72)	(1.08)
0.75	0.43	19.76	16.20	16.42	28.15	12.37	14.30	16.37	36.11	63.62	17.57	17.14	17.73	36.83	64.27
		(0.12)	(0.07)	(0.11)	(0.26)	(0.09)	(0.10)	(0.12)	(0.35)	(0.63)	(0.08)	(0.10)	(0.12)	(0.35)	(0.64)
1.00	0.58	12.26	11.45	10.14	14.42	7.70	8.70	9.59	18.87	35.90	12.44	11.27	11.22	19.42	35.79
		(0.07)	(0.04)	(0.05)	(0.12)	(0.05)	(0.06)	(0.06)	(0.17)	(0.35)	(0.04)	(0.05)	(0.06)	(0.17)	(0.35)
1.25	0.72	8.45	8.84	7.33	8.55	5.40	6.01	6.41	10.89	21.19	9.66	8.45	7.98	11.19	21.35
		(0.04)	(0.03)	(0.03)	(0.06)	(0.03)	(0.04)	(0.04)	(0.09)	(0.20)	(0.03)	(0.03)	(0.04)	(0.09)	(0.20)
1.50	0.87	6.27	7.27	5.75	5.72	4.02	4.47	4.77	7.16	12.65	7.90	6.74	6.29	7.41	12.88
		(0.03)	(0.02)	(0.02)	(0.04)	(0.02)	(0.03)	(0.03)	(0.05)	(0.12)	(0.02)	(0.02)	(0.02)	(0.05)	(0.12)
1.75	1.01	4.91	6.16	4.72	4.33	3.18	3.56	3.69	5.04	8.20	6.71	5.60	5.16	5.25	8.45
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.03)	(0.07)	(0.02)	(0.02)	(0.02)	(0.03)	(0.07)
2.00	1.15	3.92	5.36	4.06	3.47	2.59	2.91	3.00	3.79	5.64	5.85	4.82	4.41	4.10	5.89
		(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.02)	(0.02)	(0.02)	(0.05)	(0.01)	(0.01)	(0.01)	(0.02)	(0.05)
2.25	1.30	3.32	4.79	3.57	2.92	2.21	2.41	2.51	3.02	4.09	5.19	4.27	3.84	3.28	4.27
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.03)	(0.01)	(0.01)	(0.01)	(0.02)	(0.03)
2.50	1.44	2.78	4.32	3.20	2.53	1.90	2.06	2.16	2.50	3.16	4.67	3.83	3.43	2.76	3.25
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
2.75	1.59	2.43	3.95	2.90	2.24	1.69	1.83	1.87	2.10	2.53	4.28	3.48	3.11	2.41	2.61
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
3.00	1.73	2.12	3.63	2.66	2.02	1.49	1.62	1.68	1.82	2.09	3.93	3.20	2.84	2.13	2.14
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.25	1.88	1.89	3.38	2.46	1.85	1.37	1.47	1.52	1.65	1.79	3.65	2.95	2.62	1.90	1.84
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.50	2.02	1.69	3.17	2.31	1.71	1.27	1.33	1.38	1.48	1.56	3.41	2.76	2.44	1.74	1.60
		(0.01)	(0.01)	(0.01)	(0.01)		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.75	2.17	1.55	2.99	2.18	1.58	1.19	1.25	1.28	1.36	1.41	3.21	2.59	2.29	1.59	1.44
		(0.01)	(0.01)		(0.01)				(0.01)	(0.01)	(0.01)	(0.01)		(0.01)	(0.01)
4.00	2.31	1.42	2.83	2.08	1.47	1.13	1.18	1.20	1.26	1.28	3.04	2.43	2.17	1.46	1.32
4.25	2.45	1.32	2.68	2.00	1.37	1.08	1.11	1.14	1.18	1.19	2.89	2.29	2.08	1.36	1.22
4.50	2.60	1.23	2.53	1.95	1.29	1.05	1.08	1.08	1.12	1.13	2.76	2.17	2.01	1.28	1.15
4.75	2.74	1.16	2.38	1.88	1.21	1.03	1.05	1.06	1.08	1.08	2.61	2.10	1.96	1.20	1.10
5.00	2.89	1.11	2.27	1.82	1.14	1.02	1.03	1.03	1.04	1.05	2.47	2.05	1.91	1.13	1.06
		B = 7.94	h = 8.79	h = 5.55	h = 3.15	h4 = 9.82	h4 = 10.99	h4 = 11.57	h4 = 12.71	h4 = 12.82	h4 = 9.39	h4 = 10.82	h4 = 11.52	h4 = 12.69	h4 = 12.86
RMI:		0.41	1.06	0.64	0.51	0.00	0.09	0.16	0.58	1.22	1.23	0.87	0.75	0.70	1.26

Table B.6: $\tau = 0$, $p = 10$, $\mu_a = (\delta,\delta,\delta,\ldots,\delta)$

			MC1 CUSUM			MEWMA Actual Covariance Matrix					MEWMA Steady-State Covariance Matrix				
Distance λ	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8
0.00	0.00	200.13	200.58	201.14	198.96	203.74	202.87	200.72	201.35	200.15	202.26	195.65	199.03	201.22	202.86
		(1.92)	(2.06)	(2.05)	(1.94)	(2.19)	(2.07)	(2.03)	(2.01)	(1.97)	(1.84)	(1.84)	(1.90)	(1.99)	(2.09)
0.25	0.08	119.83	85.63	120.92	160.33	92.45	116.94	129.88	171.29	182.87	99.98	112.68	131.13	174.69	189.89
		(0.94)	(0.81)	(1.20)	(1.61)	(0.91)	(1.17)	(1.29)	(1.71)	(1.80)	(0.79)	(1.02)	(1.25)	(1.73)	(1.86)
0.50	0.16	51.82	33.65	44.42	88.95	34.44	44.51	55.56	115.14	144.10	42.75	47.18	57.98	114.15	147.00
		(0.33)	(0.17)	(0.41)	(0.89)	(0.28)	(0.38)	(0.50)	(1.14)	(1.42)	(0.24)	(0.36)	(0.49)	(1.11)	(1.47)
0.75	0.24	27.54	21.60	19.81	41.49	18.01	21.59	25.91	66.73	102.74	24.79	24.31	27.55	66.01	104.12
		(0.16)	(0.08)	(0.13)	(0.41)	(0.13)	(0.16)	(0.21)	(0.64)	(1.04)	(0.10)	(0.14)	(0.20)	(0.64)	(1.03)
1.00	0.32	17.25	16.05	12.67	19.19	11.11	12.89	14.37	37.34	67.92	17.45	15.84	16.47	37.64	68.83
		(0.09)	(0.05)	(0.06)	(0.18)	(0.07)	(0.08)	(0.10)	(0.36)	(0.67)	(0.06)	(0.07)	(0.10)	(0.35)	(0.69)
1.25	0.40	11.92	12.93	9.51	10.55	7.71	8.70	9.45	21.36	42.96	13.54	11.63	11.32	21.89	44.72
		(0.06)	(0.03)	(0.03)	(0.08)	(0.05)	(0.05)	(0.06)	(0.19)	(0.41)	(0.04)	(0.04)	(0.05)	(0.19)	(0.43)
1.50	0.47	8.73	10.79	7.72	7.06	5.66	6.38	6.72	13.09	27.51	11.08	9.19	8.59	13.53	28.11
		(0.04)	(0.02)	(0.02)	(0.04)	(0.03)	(0.04)	(0.04)	(0.11)	(0.27)	(0.03)	(0.03)	(0.03)	(0.11)	(0.27)
1.75	0.55	6.73	9.28	6.56	5.35	4.41	4.96	5.17	8.60	18.02	9.33	7.61	7.02	8.87	17.83
		(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.03)	(0.03)	(0.07)	(0.17)	(0.02)	(0.02)	(0.02)	(0.06)	(0.16)
2.00	0.63	5.38	8.18	5.68	4.38	3.59	4.02	4.19	6.15	11.78	8.15	6.54	5.88	6.48	11.91
		(0.02)	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)	(0.02)	(0.04)	(0.11)	(0.02)	(0.02)	(0.02)	(0.04)	(0.11)
2.25	0.71	4.44	7.34	5.05	3.73	3.00	3.30	3.46	4.57	8.12	7.22	5.74	5.15	4.96	8.35
		(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)	(0.03)	(0.07)	(0.01)	(0.01)	(0.01)	(0.03)	(0.07)
2.50	0.79	3.75	6.66	4.56	3.29	2.58	2.81	2.91	3.67	5.79	6.51	5.14	4.54	3.99	5.92
		(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.05)	(0.01)	(0.01)	(0.01)	(0.02)	(0.05)
2.75	0.87	3.24	6.08	4.15	2.93	2.24	2.44	2.52	3.02	4.33	5.93	4.64	4.07	3.32	4.54
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.03)	(0.01)	(0.01)	(0.01)	(0.01)	(0.03)
3.00	0.95	2.83	5.61	3.84	2.67	1.97	2.15	2.23	2.55	3.44	5.44	4.27	3.75	2.89	3.53
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
3.25	1.03	2.48	5.23	3.56	2.47	1.77	1.90	1.96	2.23	2.74	5.05	3.93	3.44	2.55	2.85
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
3.50	1.11	2.23	4.88	3.33	2.31	1.60	1.72	1.79	1.95	2.31	4.72	3.67	3.21	2.32	2.39
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.75	1.19	2.01	4.59	3.16	2.18	1.47	1.58	1.62	1.77	1.97	4.42	3.43	3.01	2.11	2.05
		(0.01)	(0.01)	(0.01)		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
4.00	1.26	1.83	4.33	2.99	2.08	1.36	1.44	1.48	1.60	1.72	4.17	3.25	2.83	1.94	1.78
4.25	1.34	1.68	4.11	2.84	1.99	1.27	1.34	1.38	1.47	1.54	3.94	3.08	2.66	1.82	1.60
4.50	1.42	1.54	3.92	2.69	1.91	1.21	1.26	1.27	1.35	1.40	3.75	2.94	2.51	1.70	1.44
4.75	1.50	1.44	3.75	2.56	1.85	1.14	1.18	1.21	1.26	1.29	3.55	2.81	2.37	1.59	1.33
5.00	1.58	1.34	3.55	2.41	1.78	1.09	1.13	1.16	1.19	1.21	3.38	2.68	2.25	1.48	1.24
		B = 14.75	h = 15.33	h = 9.58	h = 5.53	h4 = 21.41	h4 = 22.98	h4 = 23.7	h4 = 25.07	h4 = 25.16	h4 = 20.76	h4 = 22.56	h4 = 23.53	h4 = 25.06	h4 = 25.21
RMI:		0.43	1.43	0.76	0.55	0.01	0.12	0.19	0.81	1.74	1.40	0.97	0.80	0.95	1.81

Table B.7: $\tau = 50$, $p = 2$, $\mu_a = (\delta, 0)$

			MC1 CUSUM			MEWMA Actual Covariance Matrix					MEWMA Steady-State Covariance Matrix				
Distance λ	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8
0.00	0.00	195.99	192.53	195.66	197.57	208.01	202.18	199.56	206.28	203.21	202.13	202.95	203.86	195.58	202.06
		(1.97)	(1.88)	(2.05)	(2.02)	(2.33)	(2.11)	(2.04)	(2.08)	(2.03)	(2.06)	(2.06)	(2.06)	(1.94)	(2.00)
0.25	0.25	81.34	66.45	91.34	130.26	62.24	74.26	84.09	138.85	163.51	63.31	75.79	87.29	133.86	159.30
		(0.67)	(0.57)	(0.88)	(1.29)	(0.60)	(0.73)	(0.83)	(1.37)	(1.63)	(0.55)	(0.70)	(0.82)	(1.33)	(1.57)
0.50	0.50	31.36	25.98	31.19	55.63	23.60	26.36	30.36	63.29	96.32	25.26	27.23	30.42	62.63	94.57
		(0.22)	(0.17)	(0.26)	(0.54)	(0.18)	(0.21)	(0.26)	(0.61)	(0.96)	(0.17)	(0.21)	(0.25)	(0.60)	(0.94)
0.75	0.75	16.14	15.48	15.40	25.21	13.98	14.09	14.88	29.40	52.27	14.98	14.53	15.26	29.69	52.04
		(0.10)	(0.08)	(0.10)	(0.23)	(0.09)	(0.09)	(0.10)	(0.28)	(0.51)	(0.08)	(0.09)	(0.10)	(0.28)	(0.52)
1.00	1.00	10.34	11.02	9.78	12.85	9.92	9.38	9.52	15.85	28.69	10.64	9.66	9.61	15.48	28.29
		(0.06)	(0.05)	(0.06)	(0.10)	(0.06)	(0.05)	(0.06)	(0.14)	(0.28)	(0.05)	(0.05)	(0.06)	(0.14)	(0.27)
1.25	1.25	7.12	8.69	7.19	7.98	7.71	7.04	6.90	9.48	16.62	8.32	7.35	6.98	9.22	16.61
		(0.04)	(0.04)	(0.04)	(0.06)	(0.04)	(0.04)	(0.04)	(0.08)	(0.16)	(0.04)	(0.04)	(0.04)	(0.07)	(0.16)
1.50	1.50	5.31	7.13	5.71	5.57	6.29	5.68	5.40	6.28	10.33	6.77	5.85	5.47	6.18	10.16
		(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.05)	(0.09)	(0.03)	(0.03)	(0.03)	(0.05)	(0.09)
1.75	1.75	4.24	6.06	4.76	4.31	5.35	4.75	4.44	4.58	6.80	5.79	4.92	4.52	4.54	6.75
		(0.02)	(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)	(0.03)	(0.06)	(0.02)	(0.02)	(0.02)	(0.03)	(0.06)
2.00	2.00	3.42	5.31	4.12	3.49	4.66	4.12	3.80	3.57	4.72	5.00	4.24	3.86	3.52	4.70
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.04)	(0.02)	(0.02)	(0.02)	(0.02)	(0.04)
2.25	2.25	2.86	4.75	3.64	2.96	4.19	3.62	3.35	2.91	3.52	4.47	3.78	3.40	2.92	3.56
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.03)	(0.02)	(0.01)	(0.01)	(0.02)	(0.03)
2.50	2.50	2.46	4.32	3.26	2.59	3.74	3.27	3.00	2.45	2.78	4.03	3.35	3.03	2.45	2.80
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)
2.75	2.75	2.14	3.94	2.98	2.32	3.46	2.96	2.72	2.12	2.27	3.69	3.06	2.76	2.13	2.25
		(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.00	3.00	1.89	3.64	2.75	2.10	3.17	2.76	2.50	1.92	1.90	3.40	2.81	2.53	1.90	1.91
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.25	3.25	1.70	3.40	2.55	1.91	2.94	2.53	2.32	1.73	1.65	3.15	2.61	2.35	1.72	1.65
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.50	3.50	1.54	3.17	2.39	1.78	2.78	2.40	2.16	1.58	1.47	2.96	2.45	2.19	1.57	1.47
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.75	3.75	1.42	2.98	2.26	1.66	2.62	2.25	2.04	1.48	1.33	2.76	2.29	2.06	1.46	1.33
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
4.00	4.00	1.31	2.83	2.16	1.55	2.47	2.12	1.93	1.37	1.23	2.63	2.19	1.95	1.37	1.23
4.25	4.25	1.23	2.68	2.05	1.46	2.36	2.03	1.83	1.29	1.16	2.50	2.06	1.84	1.27	1.16
4.50	4.50	1.17	2.56	1.96	1.40	2.25	1.92	1.76	1.22	1.11	2.38	1.98	1.76	1.20	1.10
4.75	4.75	1.11	2.45	1.87	1.34	2.14	1.86	1.68	1.16	1.06	2.29	1.89	1.68	1.15	1.06
5.00	5.00	1.06	2.36	1.79	1.29	2.08	1.77	1.61	1.11	1.04	2.19	1.82	1.63	1.11	1.04
		B = 6.66	h = 7.52	h = 4.78	h = 2.69	h4 = 7.86	h4 = 8.86	h4 = 9.39	h4 = 10.5	h4 = 10.63	h4 = 7.6	h4 = 8.77	h4 = 9.39	h4 = 10.42	h4 = 10.58
RMI:		0.07	0.76	0.42	0.30	0.55	0.39	0.31	0.31	0.66	0.65	0.42	0.33	0.29	0.65

Table B.8: $\tau = 50$, $p = 3$, $\mu_a = (\delta,0,0)$

			MC1 CUSUM			MEWMA Actual Covariance Matrix					MEWMA Steady-State Covariance Matrix				
Distance λ	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8
0.00	0.00	196.66	195.90	200.35	199.59	200.59	205.52	199.53	198.53	195.51	202.47	201.07	203.06	199.11	204.85
		(1.95)	(1.89)	(1.99)	(1.98)	(2.30)	(2.18)	(2.10)	(2.00)	(1.92)	(2.06)	(2.01)	(2.04)	(2.00)	(2.05)
0.25	0.25	90.41	73.56	100.07	141.04	65.65	85.34	98.44	147.81	165.10	71.09	85.22	99.44	145.75	169.09
		(0.74)	(0.62)	(0.96)	(1.40)	(0.65)	(0.85)	(0.97)	(1.48)	(1.64)	(0.62)	(0.81)	(0.96)	(1.46)	(1.66)
0.50	0.50	34.68	29.52	35.35	65.06	25.98	30.14	34.37	75.91	106.45	27.91	30.37	35.26	75.23	111.44
		(0.23)	(0.19)	(0.29)	(0.63)	(0.20)	(0.25)	(0.30)	(0.75)	(1.07)	(0.19)	(0.24)	(0.30)	(0.74)	(1.12)
0.75	0.75	18.44	17.90	17.80	28.73	14.98	15.72	16.90	36.15	61.68	16.41	16.11	17.16	35.74	64.13
		(0.11)	(0.10)	(0.12)	(0.26)	(0.10)	(0.11)	(0.12)	(0.34)	(0.62)	(0.09)	(0.10)	(0.12)	(0.34)	(0.65)
1.00	1.00	11.44	12.97	11.21	14.80	10.50	10.32	10.54	19.15	34.94	11.50	10.70	10.54	19.23	36.12
		(0.07)	(0.06)	(0.06)	(0.12)	(0.06)	(0.06)	(0.06)	(0.18)	(0.35)	(0.06)	(0.06)	(0.06)	(0.17)	(0.35)
1.25	1.25	7.96	10.14	8.24	9.08	8.14	7.71	7.53	11.18	20.78	8.95	7.97	7.63	11.22	21.30
		(0.04)	(0.05)	(0.04)	(0.06)	(0.05)	(0.04)	(0.04)	(0.09)	(0.20)	(0.04)	(0.04)	(0.04)	(0.09)	(0.20)
1.50	1.50	5.97	8.43	6.53	6.25	6.64	6.09	5.87	7.25	12.55	7.31	6.30	5.98	7.23	12.89
		(0.03)	(0.04)	(0.03)	(0.04)	(0.04)	(0.03)	(0.03)	(0.05)	(0.11)	(0.03)	(0.03)	(0.03)	(0.05)	(0.12)
1.75	1.75	4.64	7.21	5.52	4.79	5.68	5.12	4.81	5.14	8.19	6.22	5.25	4.88	5.26	8.42
		(0.02)	(0.03)	(0.02)	(0.03)	(0.03)	(0.02)	(0.02)	(0.03)	(0.07)	(0.03)	(0.02)	(0.02)	(0.04)	(0.07)
2.00	2.00	3.77	6.34	4.71	3.91	4.90	4.43	4.08	3.98	5.64	5.44	4.52	4.15	3.95	5.77
		(0.02)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.05)	(0.02)	(0.02)	(0.02)	(0.02)	(0.05)
2.25	2.25	3.12	5.59	4.15	3.35	4.38	3.89	3.59	3.20	4.13	4.80	3.99	3.64	3.20	4.23
		(0.01)	(0.02)	(0.02)	(0.01)	(0.02)	(0.02)	(0.01)	(0.02)	(0.03)	(0.02)	(0.02)	(0.01)	(0.02)	(0.03)
2.50	2.50	2.69	5.06	3.78	2.93	3.98	3.47	3.15	2.69	3.16	4.33	3.57	3.23	2.68	3.21
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)
2.75	2.75	2.33	4.67	3.41	2.62	3.60	3.19	2.86	2.33	2.55	3.93	3.25	2.91	2.32	2.57
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
3.00	3.00	2.06	4.26	3.16	2.37	3.35	2.92	2.65	2.05	2.09	3.62	3.00	2.69	2.06	2.16
		(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.25	3.25	1.84	3.96	2.92	2.16	3.11	2.71	2.45	1.85	1.82	3.36	2.77	2.48	1.86	1.85
		(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.50	3.50	1.66	3.72	2.74	2.02	2.91	2.51	2.27	1.70	1.60	3.13	2.60	2.32	1.69	1.60
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.75	3.75	1.51	3.47	2.57	1.89	2.74	2.39	2.14	1.56	1.42	2.96	2.45	2.18	1.57	1.43
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
4.00	4.00	1.40	3.32	2.43	1.78	2.60	2.24	2.03	1.45	1.30	2.80	2.30	2.06	1.45	1.31
4.25	4.25	1.30	3.14	2.32	1.67	2.47	2.15	1.92	1.36	1.21	2.65	2.18	1.95	1.35	1.22
4.50	4.50	1.22	2.98	2.23	1.57	2.36	2.04	1.83	1.28	1.14	2.51	2.08	1.86	1.27	1.14
4.75	4.75	1.15	2.84	2.13	1.51	2.26	1.95	1.75	1.21	1.10	2.42	1.99	1.78	1.21	1.10
5.00	5.00	1.10	2.71	2.05	1.44	2.16	1.88	1.70	1.15	1.06	2.32	1.91	1.71	1.15	1.06
		B = 7.94	h = 8.79	h = 5.55	h = 3.15	h4 = 9.97	h4 = 11.11	h4 = 11.62	h4 = 12.70	h4 = 12.78	h4 = 9.66	h4 = 10.94	h4 = 11.58	h4 = 12.71	h4 = 12.86
RMI:		0.08	0.92	0.51	0.37	0.53	0.39	0.31	0.35	0.75	0.65	0.42	0.33	0.35	0.79

Table B.9: τ = 50, p = 2, μ_a = (δ,0,0,…,0)

			MC1 CUSUM			MEWMA Actual Covariance Matrix					MEWMA Steady-State Covariance Matrix				
Distance λ	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8
0.00	0.00	197.14	199.09	204.68	202.99	208.78	202.92	199.05	194.39	198.48	194.43	202.96	194.23	201.47	203.91
		(2.02)	(2.06)	(2.05)	(1.94)	(2.58)	(2.29)	(2.08)	(1.93)	(1.97)	(2.00)	(2.05)	(1.98)	(1.99)	(2.03)
0.25	0.25	113.42	105.73	127.87	159.73	88.63	112.94	130.05	168.70	185.43	93.57	117.30	126.63	172.52	187.21
		(0.92)	(0.91)	(1.22)	(1.57)	(1.02)	(1.23)	(1.37)	(1.69)	(1.86)	(0.85)	(1.12)	(1.24)	(1.67)	(1.87)
0.50	0.50	48.09	48.30	52.38	90.15	33.38	44.01	54.99	113.78	150.13	38.18	46.59	54.84	114.79	150.44
		(0.32)	(0.31)	(0.43)	(0.88)	(0.31)	(0.43)	(0.53)	(1.13)	(1.50)	(0.27)	(0.39)	(0.49)	(1.12)	(1.52)
0.75	0.75	25.27	29.72	27.01	43.13	18.69	21.23	25.74	64.35	105.49	21.76	23.07	26.15	65.79	104.96
		(0.15)	(0.16)	(0.18)	(0.39)	(0.15)	(0.17)	(0.21)	(0.63)	(1.03)	(0.12)	(0.16)	(0.20)	(0.65)	(1.03)
1.00	1.00	15.86	21.79	18.22	22.41	12.97	13.39	14.83	36.81	69.62	14.98	14.51	15.42	37.19	70.30
		(0.09)	(0.11)	(0.10)	(0.18)	(0.09)	(0.09)	(0.10)	(0.35)	(0.69)	(0.07)	(0.08)	(0.10)	(0.35)	(0.69)
1.25	1.25	11.11	17.25	13.79	13.40	9.89	9.54	10.18	21.09	43.96	11.45	10.45	10.42	21.30	44.93
		(0.06)	(0.08)	(0.07)	(0.09)	(0.06)	(0.06)	(0.06)	(0.19)	(0.42)	(0.05)	(0.05)	(0.06)	(0.19)	(0.44)
1.50	1.50	8.16	14.23	11.06	9.59	7.94	7.59	7.68	13.07	28.12	9.28	8.22	7.83	13.07	28.15
		(0.04)	(0.06)	(0.05)	(0.06)	(0.05)	(0.04)	(0.04)	(0.11)	(0.27)	(0.04)	(0.04)	(0.04)	(0.11)	(0.27)
1.75	1.75	6.33	12.12	9.37	7.48	6.80	6.20	6.16	8.60	17.82	7.78	6.77	6.37	8.62	18.03
		(0.03)	(0.05)	(0.04)	(0.04)	(0.04)	(0.03)	(0.03)	(0.07)	(0.17)	(0.03)	(0.03)	(0.03)	(0.07)	(0.17)
2.00	2.00	5.12	10.55	8.13	6.24	5.82	5.29	5.13	6.28	11.87	6.76	5.79	5.28	6.24	11.89
		(0.02)	(0.04)	(0.04)	(0.03)	(0.03)	(0.03)	(0.02)	(0.04)	(0.11)	(0.03)	(0.02)	(0.02)	(0.04)	(0.11)
2.25	2.25	4.28	9.41	7.12	5.32	5.13	4.61	4.43	4.75	8.24	6.00	5.04	4.60	4.77	8.21
		(0.02)	(0.04)	(0.03)	(0.02)	(0.03)	(0.02)	(0.02)	(0.03)	(0.07)	(0.02)	(0.02)	(0.02)	(0.03)	(0.07)
2.50	2.50	3.60	8.49	6.44	4.70	4.68	4.15	3.92	3.79	5.87	5.35	4.49	4.07	3.82	5.96
		(0.01)	(0.03)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.05)	(0.02)	(0.02)	(0.01)	(0.02)	(0.05)
2.75	2.75	3.10	7.68	5.82	4.20	4.25	3.71	3.51	3.18	4.43	4.85	4.03	3.61	3.18	4.43
		(0.01)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.01)	(0.02)	(0.03)	(0.02)	(0.01)	(0.01)	(0.02)	(0.03)
3.00	3.00	2.71	7.06	5.35	3.79	3.92	3.43	3.21	2.73	3.46	4.45	3.69	3.30	2.74	3.51
		(0.01)	(0.03)	(0.02)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)
3.25	3.25	2.43	6.56	4.96	3.48	3.62	3.10	2.93	2.39	2.81	4.15	3.41	3.03	2.42	2.81
		(0.01)	(0.03)	(0.02)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
3.50	3.50	2.18	6.10	4.60	3.21	3.42	2.94	2.74	2.15	2.34	3.84	3.17	2.81	2.17	2.35
		(0.01)	(0.02)	(0.02)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.75	3.75	1.98	5.75	4.31	3.03	3.19	2.74	2.55	1.97	2.02	3.63	2.94	2.61	1.97	2.00
		(0.01)	(0.02)	(0.02)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
4.00	4.00	1.80	5.38	4.04	2.83	3.01	2.59	2.40	1.81	1.77	3.40	2.80	2.47	1.82	1.76
4.25	4.25	1.65	5.12	3.84	2.67	2.87	2.46	2.27	1.68	1.57	3.21	2.61	2.32	1.70	1.58
4.50	4.50	1.53	4.85	3.60	2.53	2.73	2.33	2.15	1.57	1.43	3.04	2.51	2.20	1.57	1.44
4.75	4.75	1.42	4.60	3.44	2.43	2.61	2.23	2.06	1.48	1.33	2.90	2.37	2.11	1.48	1.32
5.00	5.00	1.33	4.40	3.30	2.31	2.50	2.15	1.97	1.39	1.23	2.77	2.29	2.01	1.39	1.23
		B = 14.75	h = 15.33	h = 9.58	h = 5.53	h4 = 21.97	h4 = 23.32	h4 = 23.89	h4 = 25.00	h4 = 25.20	h4 = 21.06	h4 = 22.92	h4 = 23.59	h4 = 25.04	h4 = 25.22
RMI:		0.09	1.43	0.91	0.61	0.39	0.28	0.27	0.54	1.28	0.58	0.38	0.30	0.55	1.30

Table B.10: $\tau = 0$, $p = 2$, $\mu_a = (\delta,\delta)$

			MC1 CUSUM			MEWMA Actual Covariance Matrix					MEWMA Steady-State Covariance Matrix				
Distance λ	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8
0.00	0.00	195.99	190.87	197.64	195.69	204.67	203.45	200.95	206.63	205.63	203.56	204.38	199.67	196.60	200.73
		(1.97)	(1.88)	(2.05)	(2.02)	(2.28)	(2.13)	(2.03)	(2.04)	(2.04)	(2.05)	(2.08)	(2.02)	(1.97)	(2.03)
0.25	0.18	83.17	65.93	90.19	128.61	62.02	74.49	83.54	140.23	160.72	63.75	75.16	89.64	133.01	160.75
		(1.94)	(0.55)	(0.86)	(1.27)	(0.59)	(0.73)	(0.83)	(1.37)	(1.58)	(0.56)	(0.70)	(0.87)	(1.30)	(1.60)
0.50	0.35	31.16	25.71	31.42	55.89	23.97	26.80	30.34	64.55	97.10	25.18	27.14	30.63	61.80	96.03
		(0.67)	(0.17)	(0.26)	(0.54)	(0.18)	(0.22)	(0.26)	(0.63)	(0.96)	(0.17)	(0.20)	(0.25)	(0.61)	(0.95)
0.75	0.53	16.39	15.50	15.42	25.15	14.09	14.03	15.11	30.18	51.58	14.99	14.58	15.15	29.22	52.11
		(0.21)	(0.08)	(0.10)	(0.23)	(0.09)	(0.09)	(0.11)	(0.28)	(0.51)	(0.08)	(0.09)	(0.10)	(0.27)	(0.52)
1.00	0.71	10.27	11.07	9.80	13.06	9.81	9.47	9.39	15.76	28.78	10.63	9.78	9.67	15.07	28.54
		(0.10)	(0.05)	(0.06)	(0.11)	(0.06)	(0.05)	(0.06)	(0.14)	(0.28)	(0.05)	(0.05)	(0.06)	(0.13)	(0.28)
1.25	0.88	7.20	8.61	7.22	8.06	7.78	7.06	6.86	9.39	16.62	8.27	7.30	6.97	9.07	16.24
		(0.06)	(0.04)	(0.04)	(0.06)	(0.04)	(0.04)	(0.04)	(0.08)	(0.16)	(0.04)	(0.03)	(0.04)	(0.07)	(0.15)
1.50	1.06	5.39	7.14	5.69	5.58	6.32	5.68	5.38	6.23	10.41	6.80	5.87	5.46	6.22	10.13
		(0.04)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.05)	(0.09)	(0.03)	(0.03)	(0.03)	(0.04)	(0.09)
1.75	1.24	4.20	6.11	4.78	4.29	5.40	4.78	4.45	4.53	6.75	5.78	4.93	4.50	4.52	6.73
		(0.03)	(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)	(0.03)	(0.06)	(0.02)	(0.02)	(0.02)	(0.03)	(0.06)
2.00	1.41	3.43	5.29	4.12	3.48	4.70	4.08	3.82	3.56	4.72	5.06	4.23	3.87	3.51	4.72
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.04)	(0.02)	(0.02)	(0.02)	(0.02)	(0.04)
2.25	1.59	2.85	4.78	3.64	2.96	4.17	3.64	3.32	2.89	3.52	4.48	3.73	3.38	2.89	3.52
		(0.02)	(0.02)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.02)	(0.03)	(0.02)	(0.01)	(0.01)	(0.01)	(0.03)
2.50	1.77	2.44	4.31	3.26	2.60	3.77	3.28	2.98	2.46	2.80	4.04	3.37	3.03	2.45	2.78
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)
2.75	1.94	2.14	3.95	2.97	2.33	3.42	2.98	2.71	2.17	2.25	3.68	3.05	2.76	2.13	2.25
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.00	2.12	1.90	3.66	2.75	2.10	3.18	2.73	2.48	1.93	1.91	3.39	2.81	2.53	1.90	1.90
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.25	2.30	1.71	3.40	2.56	1.95	2.96	2.55	2.29	1.72	1.66	3.16	2.60	2.35	1.72	1.67
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.50	2.47	1.55	3.20	2.39	1.78	2.79	2.37	2.16	1.59	1.49	2.93	2.44	2.19	1.59	1.46
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.75	2.65	1.42	3.01	2.26	1.66	2.61	2.26	2.04	1.48	1.33	2.77	2.30	2.06	1.48	1.33
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
4.00	2.83	1.31	2.83	2.15	1.56	2.47	2.14	1.93	1.37	1.24	2.62	2.17	1.95	1.37	1.24
4.25	3.01	1.23	2.68	2.05	1.46	2.37	2.04	1.83	1.29	1.16	2.49	2.07	1.84	1.28	1.16
4.50	3.18	1.16	2.56	1.95	1.39	2.27	1.93	1.75	1.21	1.10	2.38	1.97	1.77	1.21	1.10
4.75	3.36	1.10	2.45	1.87	1.33	2.17	1.86	1.68	1.16	1.06	2.28	1.88	1.70	1.15	1.07
5.00	3.54	1.07	2.36	1.80	1.29	2.07	1.77	1.60	1.11	1.04	2.19	1.82	1.62	1.11	1.04
		B = 6.66	h = 7.52	h = 4.78	h = 2.69	h4 = 7.86	h4 = 8.86	h4 = 9.39	h4 = 10.5	h4 = 10.63	h4 = 7.6	h4 = 8.77	h4 = 9.39	h4 = 10.42	h4 = 10.58
RMI:		0.07	0.76	0.42	0.30	0.56	0.39	0.30	0.31	0.66	0.65	0.42	0.33	0.29	0.65

Table B.11: $\tau = 0$, $p = 3$, $\mu_a = (\delta,\delta,\delta)$

			MC1 CUSUM			MEWMA Actual Covariance Matrix					MEWMA Steady-State Covariance Matrix				
Distance λ	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8
0.00	0.00	196.66	195.81	199.87	199.73	198.18	207.20	200.99	203.86	191.81	199.84	201.85	204.75	203.42	199.57
		(1.95)	(1.89)	(1.99)	(1.98)	(2.24)	(2.17)	(2.04)	(2.04)	(1.91)	(2.08)	(2.00)	(2.06)	(2.04)	(1.98)
0.25	0.14	89.82	73.56	100.07	141.04	66.64	84.19	96.97	147.92	167.33	70.64	84.02	98.85	145.63	171.08
		(0.74)	(0.62)	(0.96)	(1.40)	(0.67)	(0.83)	(0.94)	(1.48)	(1.69)	(0.61)	(0.81)	(0.96)	(1.46)	(1.74)
0.50	0.29	34.96	29.52	35.35	65.06	25.97	29.72	34.97	75.07	107.43	27.81	30.83	35.16	73.35	110.88
		(0.24)	(0.19)	(0.29)	(0.63)	(0.20)	(0.24)	(0.30)	(0.72)	(1.06)	(0.19)	(0.24)	(0.29)	(0.73)	(1.11)
0.75	0.43	18.14	17.90	17.80	28.73	14.98	16.01	16.94	36.26	61.06	16.52	16.02	17.18	36.18	63.56
		(0.11)	(0.10)	(0.12)	(0.26)	(0.10)	(0.11)	(0.12)	(0.34)	(0.60)	(0.09)	(0.10)	(0.12)	(0.34)	(0.64)
1.00	0.58	11.27	12.97	11.21	14.80	10.49	10.40	10.45	18.88	35.38	11.53	10.64	10.71	19.10	36.05
		(0.06)	(0.06)	(0.06)	(0.12)	(0.06)	(0.06)	(0.06)	(0.17)	(0.35)	(0.06)	(0.06)	(0.06)	(0.17)	(0.35)
1.25	0.72	7.94	10.14	8.24	9.08	8.13	7.71	7.47	11.13	20.31	9.00	7.91	7.68	11.28	21.03
		(0.04)	(0.05)	(0.04)	(0.06)	(0.05)	(0.04)	(0.04)	(0.09)	(0.20)	(0.04)	(0.04)	(0.04)	(0.09)	(0.20)
1.50	0.87	5.97	8.43	6.53	6.25	6.69	6.12	5.86	7.21	12.61	7.34	6.35	5.94	7.33	12.79
		(0.03)	(0.04)	(0.03)	(0.04)	(0.04)	(0.03)	(0.03)	(0.05)	(0.12)	(0.03)	(0.03)	(0.03)	(0.06)	(0.12)
1.75	1.01	4.63	7.21	5.52	4.79	5.65	5.11	4.80	5.21	8.21	6.18	5.29	4.87	5.22	8.39
		(0.02)	(0.03)	(0.02)	(0.03)	(0.03)	(0.02)	(0.02)	(0.04)	(0.07)	(0.03)	(0.02)	(0.02)	(0.03)	(0.07)
2.00	1.15	3.76	6.34	4.71	3.91	4.91	4.38	4.08	3.99	5.70	5.39	4.55	4.13	3.98	5.76
		(0.02)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.05)	(0.02)	(0.02)	(0.02)	(0.02)	(0.05)
2.25	1.30	3.15	5.59	4.15	3.35	4.37	3.89	3.55	3.18	4.12	4.79	3.99	3.62	3.22	4.24
		(0.01)	(0.02)	(0.02)	(0.01)	(0.02)	(0.02)	(0.01)	(0.02)	(0.03)	(0.02)	(0.02)	(0.01)	(0.02)	(0.03)
2.50	1.44	2.68	5.06	3.78	2.93	3.95	3.49	3.19	2.67	3.18	4.31	3.61	3.21	2.71	3.21
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)
2.75	1.59	2.32	4.67	3.41	2.62	3.59	3.17	2.88	2.33	2.53	3.94	3.28	2.93	2.32	2.60
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
3.00	1.73	2.06	4.26	3.16	2.37	3.32	2.91	2.66	2.07	2.14	3.62	2.98	2.69	2.06	2.13
		(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.25	1.88	1.85	3.96	2.92	2.16	3.09	2.71	2.44	1.86	1.82	3.37	2.77	2.47	1.84	1.84
		(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.50	2.02	1.67	3.72	2.74	2.02	2.90	2.54	2.28	1.70	1.59	3.15	2.59	2.32	1.69	1.60
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.75	2.17	1.50	3.47	2.57	1.89	2.72	2.38	2.15	1.56	1.43	2.96	2.44	2.18	1.57	1.45
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
4.00	2.31	1.40	3.32	2.43	1.78	2.59	2.25	2.04	1.46	1.32	2.80	2.30	2.04	1.45	1.31
4.25	2.45	1.30	3.14	2.32	1.67	2.47	2.13	1.92	1.36	1.22	2.64	2.19	1.95	1.36	1.22
4.50	2.60	1.21	2.98	2.23	1.57	2.35	2.03	1.84	1.27	1.15	2.51	2.07	1.86	1.28	1.15
4.75	2.74	1.15	2.84	2.13	1.51	2.26	1.95	1.76	1.22	1.10	2.40	1.99	1.78	1.20	1.10
5.00	2.89	1.10	2.71	2.05	1.44	2.15	1.88	1.69	1.16	1.06	2.32	1.91	1.71	1.15	1.06
		B = 7.94	h = 8.79	h = 5.55	h = 3.15	h4 = 9.97	h4 = 11.11	h4 = 11.62	h4 = 12.70	h4 = 12.78	h4 = 9.66	h4 = 10.94	h4 = 11.58	h4 = 12.71	h4 = 12.86
RMI:		0.07	0.91	0.51	0.37	0.52	0.38	0.31	0.35	0.75	0.65	0.42	0.33	0.35	0.78

Table B.12: $\tau = 0$, $p = 10$, $\mu_a = (\delta,\delta,\delta,\ldots,\delta)$

			MC1 CUSUM			MEWMA Actual Covariance Matrix					MEWMA Steady-State Covariance Matrix				
Distance λ	δ	MMRC	k = 0.25	k = 0.50	k = 1.00	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8	r = 0.05	r = 0.1	r = 0.15	r = 0.5	r = 0.8
0.00	0.00	197.14	202.12	205.45	202.83	198.32	202.91	203.95	196.20	202.94	193.39	203.87	193.93	196.86	201.35
		(2.02)	(2.06)	(2.05)	(1.94)	(2.47)	(2.30)	(2.13)	(1.95)	(2.01)	(1.99)	(2.04)	(1.95)	(1.96)	(2.00)
0.25	0.08	112.97	105.35	126.65	159.95	88.31	112.31	130.11	169.15	185.75	92.32	116.49	128.58	172.32	191.25
		(0.94)	(0.91)	(1.22)	(1.62)	(1.00)	(1.26)	(1.34)	(1.67)	(1.86)	(0.85)	(1.12)	(1.27)	(1.69)	(1.95)
0.50	0.16	47.67	48.31	52.22	89.59	33.94	43.65	54.49	114.37	147.77	38.15	46.78	54.84	114.14	147.44
		(0.31)	(0.32)	(0.43)	(0.88)	(0.31)	(0.42)	(0.52)	(1.12)	(1.47)	(0.27)	(0.39)	(0.50)	(1.11)	(1.46)
0.75	0.24	25.30	30.05	27.26	43.75	19.10	21.48	25.71	65.12	105.19	21.72	23.12	25.93	66.80	105.54
		(0.15)	(0.17)	(0.18)	(0.41)	(0.15)	(0.17)	(0.21)	(0.64)	(1.05)	(0.12)	(0.16)	(0.20)	(0.65)	(1.05)
1.00	0.32	15.80	21.86	18.21	22.61	12.78	13.25	14.74	36.54	69.13	14.99	14.71	15.32	37.03	70.59
		(0.09)	(0.11)	(0.10)	(0.18)	(0.09)	(0.09)	(0.10)	(0.35)	(0.69)	(0.07)	(0.08)	(0.10)	(0.35)	(0.69)
1.25	0.40	11.22	17.16	13.77	13.61	9.90	9.60	10.00	20.89	44.15	11.49	10.51	10.42	21.16	44.00
		(0.06)	(0.08)	(0.07)	(0.09)	(0.06)	(0.06)	(0.06)	(0.19)	(0.43)	(0.05)	(0.05)	(0.06)	(0.19)	(0.43)
1.50	0.47	8.18	14.28	11.15	9.64	7.98	7.44	7.50	12.97	27.55	9.26	8.23	7.88	13.33	28.00
		(0.04)	(0.06)	(0.05)	(0.06)	(0.05)	(0.04)	(0.04)	(0.11)	(0.26)	(0.04)	(0.04)	(0.04)	(0.11)	(0.27)
1.75	0.55	6.38	12.17	9.44	7.49	6.72	6.20	6.10	8.80	17.94	7.81	6.83	6.32	8.63	17.72
		(0.03)	(0.05)	(0.04)	(0.04)	(0.04)	(0.03)	(0.03)	(0.07)	(0.17)	(0.03)	(0.03)	(0.03)	(0.07)	(0.17)
2.00	0.63	5.13	10.62	8.06	6.23	5.87	5.30	5.15	6.13	11.90	6.79	5.78	5.30	6.30	11.88
		(0.02)	(0.04)	(0.04)	(0.03)	(0.03)	(0.03)	(0.02)	(0.04)	(0.11)	(0.03)	(0.02)	(0.02)	(0.04)	(0.11)
2.25	0.71	4.24	9.40	7.11	5.39	5.16	4.61	4.43	4.72	8.35	5.98	5.07	4.58	4.77	8.38
		(0.02)	(0.04)	(0.03)	(0.03)	(0.03)	(0.02)	(0.02)	(0.03)	(0.07)	(0.02)	(0.02)	(0.02)	(0.03)	(0.07)
2.50	0.79	3.60	8.48	6.43	4.68	4.60	4.12	3.90	3.81	5.86	5.36	4.47	4.07	3.85	5.82
		(0.02)	(0.03)	(0.03)	(0.02)	(0.03)	(0.02)	(0.02)	(0.02)	(0.05)	(0.02)	(0.02)	(0.01)	(0.02)	(0.05)
2.75	0.87	3.08	7.70	5.77	4.22	4.24	3.73	3.51	3.14	4.52	4.86	4.03	3.61	3.20	4.49
		(0.01)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.01)	(0.02)	(0.03)	(0.02)	(0.01)	(0.01)	(0.02)	(0.03)
3.00	0.95	2.74	7.13	5.36	3.78	3.95	3.38	3.20	2.75	3.49	4.49	3.70	3.32	2.74	3.45
		(0.01)	(0.03)	(0.02)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)
3.25	1.03	2.40	6.57	4.97	3.48	3.65	3.15	2.95	2.42	2.85	4.13	3.40	3.03	2.40	2.82
		(0.01)	(0.03)	(0.02)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
3.50	1.11	2.17	6.15	4.60	3.23	3.36	2.91	2.74	2.16	2.36	3.85	3.15	2.81	2.17	2.36
		(0.01)	(0.02)	(0.02)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3.75	1.19	1.97	5.71	4.31	3.02	3.17	2.76	2.54	1.97	2.01	3.59	2.98	2.63	1.99	2.03
		(0.01)	(0.02)	(0.02)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
4.00	1.26	1.80	5.40	4.09	2.85	2.99	2.58	2.39	1.80	1.76	3.42	2.80	2.47	1.81	1.77
4.25	1.34	1.65	5.10	3.84	2.67	2.88	2.47	2.27	1.67	1.57	3.23	2.63	2.33	1.68	1.58
4.50	1.42	1.52	4.84	3.63	2.54	2.74	2.34	2.15	1.57	1.43	3.06	2.51	2.21	1.59	1.43
4.75	1.50	1.42	4.62	3.49	2.42	2.61	2.24	2.06	1.48	1.33	2.91	2.39	2.10	1.47	1.32
5.00	1.58	1.33	4.39	3.29	2.31	2.49	2.14	1.95	1.39	1.23	2.79	2.27	2.00	1.39	1.23
		B = 14.75	h = 15.33	h = 9.58	h = 5.53	h4 = 21.97	h4 = 23.32	h4 = 23.89	h4 = 25.00	h4 = 25.20	h4 = 21.06	h4 = 22.92	h4 = 23.59	h4 = 25.04	h4 = 25.22
RMI:		0.09	1.43	0.91	0.61	0.39	0.28	0.26	0.54	1.28	0.58	0.38	0.30	0.55	1.29

Bibliography

1. Crosier, Robert B. "Multivariate Generalizations of Cumulative Sum Quality-Control Schemes". *Technometrics*, 30: 291-303, August 1988.

2. Hawkins, Douglas M., Qiu, Peihua and Kang, Chang W. "The Changepoint Model for Statistical Process Control". *Technometrics*, 35: 355-366, 2003.

3. Han, Dong and Tsung, Fugee. "A Generalized EWMA Control Chart and Its Comparison with the Optimal EWMA, CUSUM and GLR Schemes". The Annals of Statistics, 32: 316-338, 2004.

4. Han, Dong and Tsung, Fugee. "A Reference-Free Cuscore Chart for Dynamic Mean Change Detection and a Unified Framework for Charting Performance Comparison" *Journal of the American Statistical Association*, 101: 368-386, 2006.

5. Hotelling, Harold. "Multivariate Quality Control – Illustrated by the Air Testing of Bombsights". *Techniques of Statistical Analysis*

6. Lai, Tze L. "Sequential Changepoint Detection in Quality Control and Dynamical System". *Journal of the Royal Statistical Society*, 57: 613-658, 1995.

7. Lowry, Cynthia A. and Montgomery, Douglas C. "A Review of Multivariate Control Charts". *IIE Transactions*, 27: 800-810, 1995.

8. Lowry, Cynthia A., Woodall, William H., Champ, Charles W. and Rigdon, Stephen E. "A Multivariate Exponentially Weighted Moving Average Control Chart". *Technometrics*, 34: 46-53, February 1992.

9. Page, E. S. "Continuous Inspection Schemes". *Biometrika*, 41: 100-114, June 1954.

10. Page, E. S. "Cumulative Sum Charts". *Technometrics*, 3: 1-9, February 1961.

11. Pignatiello, Joseph J. and Runger, George C. "Comparisons of Multivariate CUSUM Charts". *Journal of Quality Technology*, 22: 173-186, July 1990.

12. Pignatiello, Joseph J. and Simpson, James R. "A Magnitude-Robust Control Chart for Monitoring and Estimating Step Changes for Normal Process Means". *Quality and Reliability Engineering International*, 18: 429-441, August 2002.

13. Roberts, S.W. "Control Chart Tests Based on Geometric Moving Averages". *Technometrics*, 1: 239-251, August 1959.

14. Runger, George C. and Prabhu, Sharad S. "A Markov Chain Model for the Multivariate Exponentially Weighted Moving Averages Control Chart". *Journal of the American Statistical Association*, 91: 1701-1706, December 1996.

15. Ryan, Thomas P. *Statistical Methods for Quality Control.* New York: Wiley, 2000.

16. Shewhart, Walter A. "The Application of Statistics as an Aid in Maintaining Quality of a Manufactured Product" *Journal of the American Statistical Association*, 20: 546-548, 1925.

17. Shewhart, Walter A. *Economic Control of Quality of Manufactured Product*. New York: Van Nostrand, 1931.

18. Woodall, William H. and Ncube, Matoteng M. "Multivariate CUSUM Quality-Control Procedures". *Technometrics*, 27: 285-291, August 1985.

19. Wu, Yanhong "Design of Control Charts for Detecting the Change Point". In *Change-Point Problems* (Carlstein, Edward G., Müller, Hans-Georg, Siegmund, David, eds.) 330-345. Hayward, CA: IMS, 2004.

20. Zamba, K. D. and Hawkins, Douglas M. "A Multivariate Change-Point Model for Statistical Process Control". *Technometrics*, 48: 539-549, November 2006.

REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 074-0188*

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED** *(From – To)*
23-03-2007	Master's Thesis	Sep 2005 - Mar 2007

4. TITLE AND SUBTITLE A MULTIVARIATE MAGNITUDE ROBUST CONTROL CHART FOR MEAN SHIFT DETECTION AND CHANGE POINT ESTIMATION	**5a. CONTRACT NUMBER** **5b. GRANT NUMBER** **5c. PROGRAM ELEMENT NUMBER**
6. AUTHOR(S) Harrell, Ryan, M., Captain, USAF	**5d. PROJECT NUMBER** **5e. TASK NUMBER** **5f. WORK UNIT NUMBER**
7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(S) Air Force Institute of Technology Graduate School of Engineering and Management (AFIT/EN) 2950 Hobson Street, Building 642 WPAFB OH 45433-7765	**8. PERFORMING ORGANIZATION REPORT NUMBER** AFIT/GOR/ENS/07-09
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) N/A	**10. SPONSOR/MONITOR'S ACRONYM(S)** **11. SPONSOR/MONITOR'S REPORT NUMBER(S)**

12. DISTRIBUTION/AVAILABILITY STATEMENT
APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

13. SUPPLEMENTARY NOTES

14. ABSTRACT
Statistical control charts are often used to detect a change in an otherwise stable process. This process may contain several variables affecting process stability. The goal of any control chart is to detect an out-of-control state quickly and provide insight on when the process actually changed. This reduces the off-line time the quality engineer spends assigning causality. In this research, a multivariate magnitude robust chart (MMRC) was developed using a change point model and a likelihood-ratio approach. Here the process is considered in-control until one or more normally distributed process variables permanently and suddenly shifts to out-of-control, stable value. Using average run length (ARL) performance and the relative mean index (RMI), the MMRC is compared to the multivariate cumulative sum (MC1) and the multivariate exponentially weighted moving average (MEWMA). These results show the MMRC performs favorably to the MC1 and MEWMA when the process is initially in-control before shifting out-of-control. Additionally, the MMRC provides an estimate for the change point and out-of-control mean vector. This change point estimator is shown effective for medium to large sudden mean shifts.

15. SUBJECT TERMS
statistical process control; change-point estimation; multivariate control chart; likelihood-ratio; maximum likelihood estimation; Shewhart control chart; MEWMA control chart; MC1 control chart; CUSUM control chart; Monte Carlo simulation; control limits; average run length; control charts

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**			Marcus B. Perry, PhD
U	U	U	UU	114	**19b. TELEPHONE NUMBER** *(Include area code)* (937) 255-3636 x4588 e-mail: Marcus.Perry@AFIT.edu

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39-18